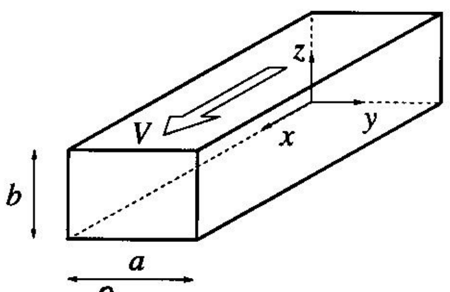
VF1 A fluid of density ρ and viscosity μ flows along an infinitely long rectangular channel of height b and width a due to the displacement of the top wall, which moves with constant longitudinal velocity V, inducing in the liquid a steady unidirectional motion with a single velocity component $v_x(y,z)$.

1. Write the conservation equation with boundary conditions that determines $v_x(y,z)$.
2. Express the problem in dimensionless form by introduction of appropriate variables $\bar{v} = v_x/V$, $\eta = y/a$ and $\xi = z/b$, demostrating that the solution is only a function of $\Lambda = a/b$.
3. Consider the limit $a \gg b$ ($\Lambda \gg 1$) and determine the approximate solution for $\bar{v}$ (note that this solution fails near the vertical bounding walls, in relatively thin regions of width $\delta y \sim b$).
4. In the limit $\Lambda \gg 1$, determine the volume flux, as well as the force acting on the upper wall and the power needed to move it (per unit length).
5. Consider the limit $a \ll b$ ($\Lambda \ll 1$), demonstrating that, in the first approximation, the fluid remains at rest in most of the channel, except in a region located near the upper wall. Write the reduce problem that determines the solution in this limiting case.
6. Using separation of variables with $\bar{v} = f(\eta)g(\xi)$, obtain the general solution of the problem stated in 1.



SEE 6)

$$\bar{v} = \sum_{m=1,\dots,\infty} \frac{4}{\pi(2m+1)} \frac{sh\left(\pi(2m+1)\frac{b}{a}\xi\right)}{sh\left(\pi(2m+1)\frac{b}{a}\right)} \sin\left(\pi(2m+1)\eta\right)$$

1) STEADY UNIDIRECTIONAL FLOW

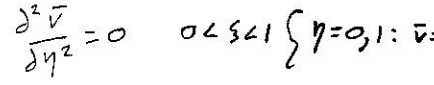
$$0 = \overset{P_\ell = 0}{} \mu\left(\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) \qquad 0<y<a \begin{cases} z=0: v_x=0 \\ z=b: v_x=V \end{cases}$$

$$0<z<b \begin{Bmatrix} y=0 \\ y=a \end{Bmatrix}: v_x = 0$$

2) DIMENSIONLESS PROBLEM

$$\frac{\partial^2 \bar{v}}{\partial \eta^2} + \Lambda^2 \frac{\partial^2 \bar{v}}{\partial \xi^2} = 0 \quad \left\{ \begin{array}{l} 0<\eta<1 \begin{cases} \xi=0: \bar{v}=0 \\ \xi=1: \bar{v}=1 \end{cases} \\ 0<\xi<1 \begin{Bmatrix} \eta=0 \\ \eta=1 \end{Bmatrix} \bar{v}=0 \end{array} \right.$$

3) $\Lambda >> 1$ THE EQUATION REDUCES TO GIVE



$$\frac{\partial^2 \bar{v}}{\partial \xi^2} = 0 \rightarrow \bar{v} = \xi \Rightarrow \left(\frac{v_x}{V} = \frac{z}{b}\right)$$

COUETTE SOLUTION.
IT DOES TO SATISFY THE BOUNDARY CONDITIONS ON THE SIDE WALLS $\eta = 0$ & $\eta = 1$
IN THIS BOUNDARY REGION ONE SHOULD USE A STRETCHED COORDINATE $\left(\bar{\eta} = \frac{y}{b} = \Lambda\eta \text{ AT } y=0\right)$

$$\frac{\partial^2 \bar{v}}{\partial \bar{\eta}^2} + \frac{\partial^2 \bar{v}}{\partial \xi^2} = 0; \quad 0<\xi<1 \begin{cases} \bar{\eta}=0: \bar{v}=0 \\ \bar{\eta}\to\infty: \bar{v}\to\xi \end{cases} \quad \bar{\eta}>0 \begin{cases} \xi=0: \bar{v}=0 \\ \xi=1: \bar{v}=1 \end{cases}$$

4) VOLUME FLUX $Q = V\dfrac{ba}{2}$

FORCE $-\dfrac{\mu V}{b} a\, \bar{e}_z$, POWER $\dfrac{\mu a}{b} V^2$

5) $\Lambda << 1$

$$\frac{\partial^2 \bar{v}}{\partial \eta^2} = 0 \qquad 0<\xi<1 \left\{ \eta = 0,1: \bar{v}=0 \right. \rightarrow \boxed{\bar{v}=0}$$

IT DOES NOT SATISFY B.C. ON UPPER WALL

FOR THE DESCRIPTION, IT IS CONVENIENT TO INTRODUCE $\bar{\xi} = \dfrac{b-z}{a}$

$$\frac{\partial^2 \bar{v}}{\partial \eta^2} + \frac{\partial^2 \bar{v}}{\partial \bar{\xi}^2} = 0 \qquad 0<\eta<1: \begin{cases} \bar{\xi}=0: \bar{v}=1 \\ \bar{\xi}\to\infty: \bar{v}\to 0 \end{cases}$$

$$\bar{\xi}>0 \left\{ \eta=0,1: \bar{v}=0 \right.$$

6) $\bar{v} = f(\eta)g(\xi) \rightarrow \dfrac{f''}{f} = -\Lambda^2 \dfrac{\ddot{g}}{g} = -\lambda^2 \rightarrow f = F_1 \sin(\lambda\eta) + F_2\cos(\lambda\eta)$

$f(0) = f(1) = 0 \quad \begin{cases} F_2 = 0 \\ F_1 \sin\lambda = 0 \Rightarrow \lambda = \lambda_n = \pi, 2\pi, \dots n\pi \end{cases}$

WITH $\lambda_n = n\pi$, $\dfrac{\ddot{g}}{g} = \left(\dfrac{b}{a}\pi n\right)^2 \Rightarrow g = G_1 e^{\frac{b}{a}\pi n \xi} + G_2 e^{-\frac{b}{a}\pi n\xi} \rightarrow G_1 + G_2 = 0$

$\bar{v} = \sum A_n \sin(\pi n\eta)\, sh\left(\pi n \frac{b}{a}\xi\right)$, $\xi=1, \bar{v}=1 \Rightarrow 1 = \sum A_n \sin(\pi n \eta)\, sh\left(\pi n\frac{b}{a}\right) \rightarrow A_{1,3,5\dots} = \dfrac{4}{\pi n\, sh\left(\frac{\pi n b}{a}\right)}$, $A_{2,4,6\dots} = 0$

PROBLEM 3 (1 hour)

A perfect liquid of constant density ρ and constant viscosity μ fills the space between two infinite parallel walls separated by a distance $2h$, that move parallel to themselves in the same direction with a constant velocity U. As shown in the figure, a very thin solid plate of length $L \gg h$ is placed at the center of the channel. In the regions far upstream and far downstream from the plate the liquid moves with uniform velocity U due to the motion of the confining walls, and the modified pressure takes uniform values $P(x \to -\infty) \to P_{-\infty}$ and $P(x \to \infty) \to P_{\infty}$, respectively.

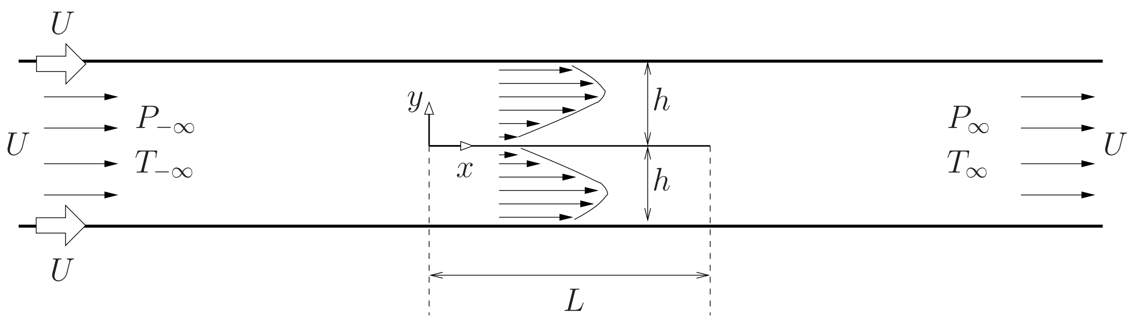
1. Provide the criterion that must be accomplished for the flow to be dominated by viscous forces. (**1 point**)

2. Obtain the velocity field in the region occupied by the plate, in terms of the unknown pressure gradient $P_l = -\partial P/\partial x$. (**3 points**)

3. Obtain the pressure drop $P_{-\infty} - P_{\infty}$, as well as the pressure gradient P_l. (**3 points**)

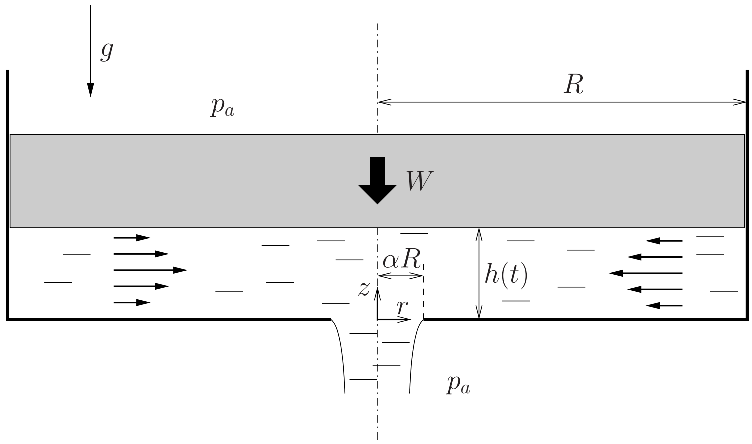
4. Determine the forces (per unit spanwise length) that the liquid exerts on the confining walls, F_w, and on the inner plate, F_p, as well as the power that must be applied to drive the walls. (**3 points**)

5. If all the inner walls are thermally isolated, determine the temperature increase in the liquid between two points situated far downstream and far upstream, $T_{\infty} - T_{-\infty}$. To that end, apply the energy equation in integral form to an appropriately defined control volume (**1 extra point**).



PROBLEM 4 (1 hour)

A cylindrical piston of weight W and radius R, whose upper side is exposed to the atmosphere, slides without friction inside a coaxial container of the same radius. The initial distance between the lower side of the piston and the bottom of the tank is $h_0 \ll R$. Below the piston, the container is filled with liquid of density ρ and viscosity μ. The piston falls due to its own weight, and forces the liquid to leave the tank, discharging to the atmosphere through a circular hole of radius $\alpha R \ll R$.



The known data are $(\rho, \mu, h_0, R, \alpha, W, g, p_a)$. The objective is to determine the vertical position of the piston as a function of time, $h(t)$, under the assumption of dominant viscous forces in the liquid. Gravitational forces in the liquid can be neglected ($P \approx p$). The following steps are suggested:

1. (*2 points*). Using the radial momentum equation, find the liquid velocity profile, $v_r(z,t)$, in the slender region $\alpha R \le r \le R$, $0 \le z \le h(t)$, as a function of $h(t)$, and of the pressure gradient $P_l = -\partial p/\partial r$.

2. (*2 points*). Obtain $p(r,t)$ as a function of $h(t)$, $\dot{h}(t) = \mathrm{d}h/\mathrm{d}t$, and known data. To that end, integrate the continuity equation across the gap (in the z direction), finding the second order differential equation satisfied by $p(r,t)$. To solve the equation, note that the pressure drop across the non-slender turning region $0 \le r \le \alpha R$, $0 \le z \le h(t)$, can be neglected, so that $p(r = \alpha R) \approx p_a$. Note also that $v_r = 0$ at the container wall $r = R$.

3. (*2 points*). Find the vertical force acting on the piston due to the liquid and the external atmosphere, F, as a function of $h(t)$, $\dot{h}(t)$, and known data. Note that it proves convenient to introduce a function $f(\alpha)$ to express the dependence of F on α.

4. (*2 points*). Using Newton's second law applied to the piston in the vertical direction, find the second-order ODE with initial conditions satisfied by $h(t)$.

5. (*2 points*). Obtain $h(t)$, as well as the volume flux through the hole, $Q(t)$, under the assumption of negligible inertia of the piston (acceleration of the piston much smaller than g). Note that, in this approximation, the pressure forces balance the weight of the piston, the ODE becomes of first order, and only the initial condition $h(0) = h_0$ can be satisfied.

6. (*2 additional points*). Provide the condition that must be accomplished for viscous forces to be dominant in the liquid, as well as the criterion of negligible inertia of the piston, in terms of the governing parameters $(\rho, \mu, h_0, R, \alpha, W, g, p_a)$.

$$\int x \ln x \, \mathrm{d}x = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right)$$

PROBLEM 4

① DOMINANT VISCOUS FORCES : $0 = \underbrace{-\frac{\partial p}{\partial r}}_{P_\ell} + \mu \frac{\partial^2 v_r}{\partial z^2}$ WITH $\begin{cases} z=0 : v_r = 0 \\ z=h : v_r = 0 \end{cases}$

GIVING POISEUILLE'S PROFILE $\boxed{v_r(z,r,t) = \frac{P_\ell(r,t)}{2\mu} z\,(h(t)-z)}$ //

② $\int_0^h \left[\frac{\partial(r v_r)}{\partial r} + \frac{\partial(r v_z)}{\partial z}\right] dz = 0 \Rightarrow \frac{\partial}{\partial r}\left[r \int_0^h v_r\, dz\right] + r\dot{h} = 0$

BUT $\int_0^h v_r\, dz = \frac{P_\ell(r,t)\, h^3}{12\mu}$ THUS $\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) = \frac{12\mu\dot h}{h^3} r \Rightarrow$

$\Rightarrow r\frac{\partial p}{\partial r} = \frac{6\mu\dot h r^2}{h^3} + A \Rightarrow \frac{\partial p}{\partial r} = \frac{6\mu\dot h r}{h^3} + \frac{A}{r}$ BUT SINCE

$\left[\frac{\partial p}{\partial r}\Big|_{r=R} = 0\right]$ $(v_r(r=R)=0) \Rightarrow A = \frac{-6\mu\dot h R^2}{h^3} \Rightarrow \frac{\partial p}{\partial r} = \frac{6\mu\dot h}{h^3}\left(r - \frac{R^2}{r}\right)$

$\Rightarrow p = \frac{6\mu\dot h}{h^3}\left(\frac{r^2}{2} - R^2 \ln r\right) + B$; $\left[p(r=\alpha R) = p_a\right] \Rightarrow$

$\Rightarrow \boxed{p - p_a = -\frac{6\mu\dot h}{h^3}\left[R^2 \ln\left(\frac{r}{\alpha R}\right) - \frac{r^2 - (\alpha R)^2}{2}\right]}$ //

③ $F = 2\pi \int_{\alpha R}^{R} (p-p_a)\, r\, dr = -\frac{12\pi\mu\dot h}{h^3} R^4 \underbrace{\int_\alpha^1 \eta\left[\ln(\eta) - \ln(\alpha) - \frac{\eta^2-\alpha^2}{2}\right] d\eta}_{f(\alpha)}$

COMPUTING THE INTEGRAL, $f(\alpha) = -\frac{1}{2}\ln(\alpha) - \frac{5}{12} + \frac{3}{4}\alpha^2 - \frac{1}{3}\alpha^3$

④ $\boxed{\begin{gathered}\frac{W}{g}\ddot h = F - W \\ h(0) = h_0 \\ \dot h(0) = 0\end{gathered}}$

⑤ IF $\ddot h \ll g \ \forall t \Rightarrow$ IN A FIRST APPROXIMATION $\boxed{F = W}$

WHENCE $\frac{dh}{dt} = -\frac{W}{12\pi\mu R^4 f(\alpha)} h^3$ WITH $h(0) = h_0$, WHOSE

SOLUTION IS GIVEN IN DIMENSIONLESS TERMS AS $\boxed{\bar h = (1+2\tau)^{-1/2}}$

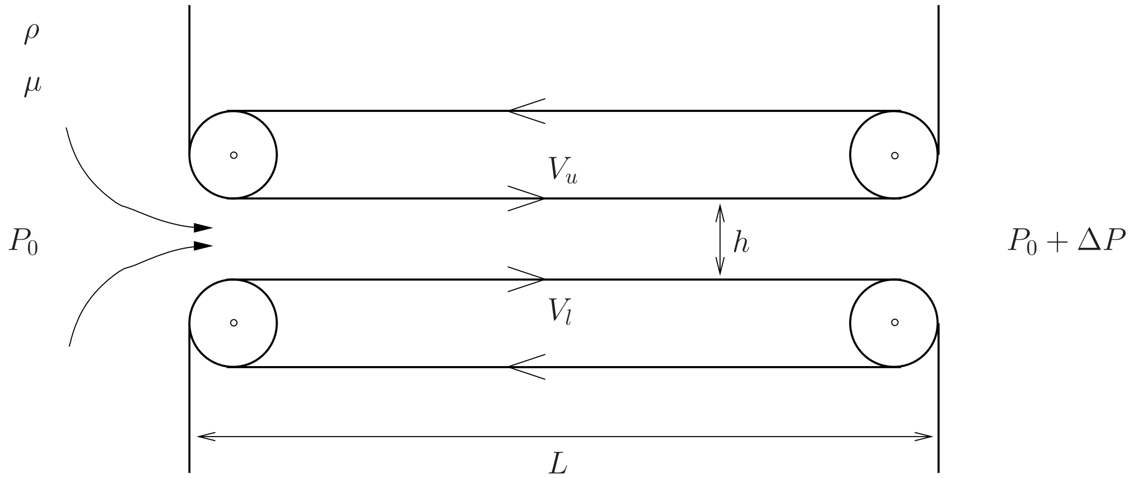
WHERE WE HAVE DEFINED $\bar h = \frac{h}{h_0}$ AND $\tau = \frac{W h_0^2}{12\pi\mu R^4 f(\alpha)} t$ (DIMENSIONLESS TIME)

⑥ VISCOUS FORCES DOMINANT IF $\frac{h_0 \dot h_c}{\nu} \lll 1$ WHERE $\dot h_c \sim \frac{h_0}{t_c}$ AND $t_c \sim \frac{\mu R^4}{W h_0^2} \Rightarrow \boxed{\frac{\rho W h_0^4}{\mu^2 R^4} \ll 1}$

INERTIA OF THE PISTON NEGLIGIBLE IF $\ddot h_c \ll g$ BUT $\ddot h_c \sim \frac{h_0}{t_c^2} \Rightarrow \boxed{\frac{W^2 h_0^5}{\mu^2 R^8 g} \ll 1}$

PROBLEM 4 (1 hour)
The system sketched in the figure is designed to steadily pump fluid of constant density ρ and constant viscosity μ from a reservoir, where the modified pressure has a constant value P_0, to another one at pressure $P_0 + \Delta P$ ($\Delta P > 0$). To that end, a pair of parallel belts of length L, separated by a distance $h \ll L$, are driven at constant velocities V_l and V_u, dragging the liquid towards the large pressure reservoir.

The liquid flow can be assumed steady, two-dimensional (spanwise length much larger than h), and dominated by viscous forces in the gap. It proves convenient to introduce a cartersian coordinate system (x, y), where x and y are the directions parallel and perpendicular to the belts, respectively. The **only** known parameters are $(\rho, \mu, P_0, \Delta P, L, h, V_l, V_u)$.



1. (*1 point*). Under the assumption of dominant viscous forces, obtain the characteristic velocity, $V_{\Delta P}$, induced in the liquid by the overpressure ΔP. Show that the pump will only work if the dimensionless parameter $\Lambda = (h^2 \Delta P)/[\mu L(V_l + V_u)]$, is sufficiently small.

2. (*1 point*). Provide the condition(s) that must be satisfied for the flow to be dominated by viscous forces.

3. (*2 points*). Determine the streamwise velocity profile in the gap between both belts, $v_x(x, y)$, as well as the volume flux per unit spanwise length, q. The results must be expressed as a function of known data, and of the reduced pressure gradient, P_l.

4. (*2 points*). Determine P_l, as well as the pressure distribution inside the channel, $P(x)$.

5. (*2 points*). Find the external forces per unit spanwise length in the x-direction, $F_{l,x}^{\mathrm{EXT}}$ and $F_{u,x}^{\mathrm{EXT}}$, that must be exerted on the lower and upper belts to keep the motion, as well as the power per unit spanwise length that needs to be supplied to drive the system, $\dot{W}$.

6. (*2 points*). Knowing that the net mechanical energy gained by the liquid per unit time and per unit spanwise length during the pumping process is $\dot{W}_{\text{useful}} = q\,\Delta P$, determine the efficiency of the pumping system, $\eta = \dot{W}_{\text{useful}}/\dot{W}$, as a function of the governing parameters. Study the particular cases a) $V_l = V_u$, and b) $V_l = 0$, demonstrating that in both cases the result can be expressed as a function of the single parameter Λ defined above. In case b), find the maximum efficiency, $\eta_{\max}$ as well as the corresponding value of $\Lambda_{\max}$.

7. (*1 additional point*). Discuss the solution of Problem 3 in the light of the results obtained in Problem 4.

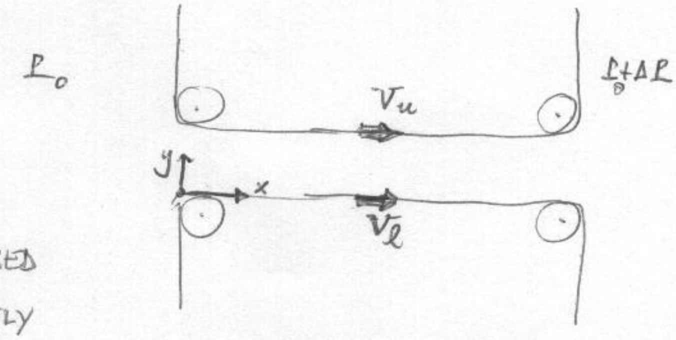
1. CRITERION FOR DOMINANT VISCOUS FORCES

↳ TWO CHARACTERISTIC VELOCITIES IN THE PROBLEM: $V_\ell + V_u$; $V_{\Delta p}$ GIVEN BY THE BALANCE

$$\frac{\Delta p}{L} \sim \mu \frac{V_{\Delta p}}{h^2} \Rightarrow V_{\Delta p} \sim \frac{h^2 \Delta p}{\mu L}$$

$$\frac{V_{\Delta p}}{V_\ell + V_u} = \frac{h^2 \Delta p}{\mu L (V_\ell + V_u)} = \Lambda$$

$$\begin{cases} \Lambda \ll 1 : \text{ CRITERION: } \boxed{\dfrac{\rho (V_\ell + V_u) h^2}{\mu L} \ll 1} \\ \Lambda \gg 1 : \boxed{\dfrac{\rho \Delta p\, h^4}{\mu^2 L^2} \ll 1} \end{cases}$$



FOR THE FLUID TO ~~MOVE~~ BE PUMPED IT IS REQUIRED THAT Λ IS SMALL ENOUGH ($V_{\Delta p}$ SUFFICIENTLY SMALLER THAN $(V_u + V_\ell)$)

2. $$\frac{\partial^2 v_x}{\partial y^2} = -\frac{p_\ell}{\mu} \Rightarrow v_x = -\frac{p_\ell y^2}{2\mu} + Ay + B$$

$y=0: \; v_x = V_\ell \Rightarrow B = V_\ell$

$y=h: \; v_x = V_u \Rightarrow A = \dfrac{V_u - V_\ell}{h} + \dfrac{p_\ell h}{2\mu}$

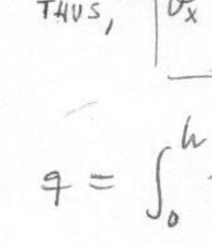
THUS, $$\boxed{v_x = V_\ell + \frac{(V_u - V_\ell)}{h} y + \frac{p_\ell}{2\mu} y(h-y)}$$ // . THE FLOW RATE P.U. SPANWISE LENGTH IS

$$q = \int_0^h v_x(y)\,dy \rightarrow \boxed{q = \frac{(V_\ell + V_u)h}{2} + \frac{p_\ell h^3}{12\mu}} \; //$$

3. $$\boxed{\frac{\partial q}{\partial x} = \int_0^h \frac{\partial v_x}{\partial x} dy = -\int_0^h \frac{\partial v_y}{\partial y} dy = v_y(0) - v_y(h) = 0} \Rightarrow \frac{\partial p_\ell}{\partial x} = 0 \Rightarrow$$

$$\Rightarrow \begin{cases} \dfrac{\partial^2 p}{\partial x^2} = 0 \\ p(x=0) = p_0 \\ p(x=L) = p_0 + \Delta p \end{cases} \Rightarrow \boxed{p = p_0 + \Delta p \frac{x}{L}} \Rightarrow \boxed{p_\ell = -\frac{\Delta p}{L}} \quad \left(p_\ell = -\frac{\partial p}{\partial x}\right)$$

4. LET $\vec{F}_u$ AND $\vec{F}_\ell$ THE FORCES P.U. SPANWISE LENGTH THAT THE LIQUID EXERTS ON THE UPPER AND LOWER BELTS, RESPECTIVELY. THEN



$$\vec{F}_u = -\int_0^L p(-\vec{e}_y)\,dx + \int_0^L \bar{\bar{\tau}}' \cdot (-\vec{e}_y)\,dx \rightarrow F_{ux} = -\int_0^L (\vec{e}_x \cdot \bar{\bar{\tau}}' \cdot \vec{e}_y)\Big|_{y=h} dx$$

$$\vec{F}_\ell = -\int_0^L p(\vec{e}_y)\,dx + \int_0^L \bar{\bar{\tau}}' \cdot \vec{e}_y\,dx \longrightarrow F_{\ell x} = \int_0^L (\vec{e}_x \cdot \bar{\bar{\tau}}' \cdot \vec{e}_y)\Big|_{y=0} dx$$

WE'LL ONLY CONSIDER THE FORCES IN THE X-DIRECTION

$$\vec{e_x} \cdot \bar{\bar{\tau}}' \cdot \vec{e_y} = \tau'_{xy} = \mu \frac{\partial v_x}{\partial y} = \mu \frac{(V_u - V_l)}{h} - \frac{\Delta P}{2L}(h-2y)$$

$$(\vec{e_x} \cdot \bar{\bar{\tau}}' \cdot \vec{e_y})\big|_{y=0} = \mu \frac{(V_u - V_l)}{h} - \frac{\Delta P h}{2L} \Rightarrow F_{lx} = \frac{\mu (V_u - V_l) L}{h} - \frac{\Delta P h}{2}$$

$$(\vec{e_x} \cdot \bar{\bar{\tau}}' \cdot \vec{e_y})\big|_{y=h} = \mu \frac{(V_u - V_l)}{h} + \frac{\Delta P h}{2} \Rightarrow F_{ux} = \frac{-\mu (V_u - V_l) L}{h} - \frac{\Delta P h}{2}$$

FORCES THAT THE FLUID EXERTS ON THE BELTS

$$\boxed{\begin{aligned} F_{ux}^{(EXT)} &= -F_{ux} = \frac{\Delta P h}{2} + \frac{\mu (V_u - V_l) L}{h} \\ F_{lx}^{(EXT)} &= -F_{lx} = \frac{\Delta P h}{2} - \frac{\mu (V_u - V_l) L}{h} \end{aligned}}$$

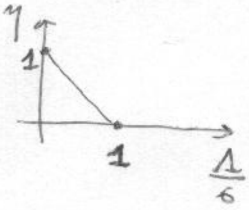
EXTERNAL FORCES NEEDED TO DRIVE THE BELTS

POWER NEEDED TO MOVE THE BELTS P.U. SPANWISE LENGTH:

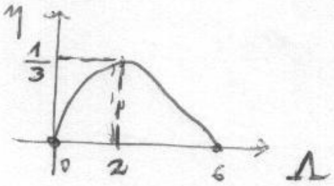
$$\dot{W} = F_{lx}^{(EXT)} V_l + F_{ux}^{(EXT)} V_u \Rightarrow \boxed{\dot{W} = \frac{(V_l + V_u) h \Delta P}{2} + \frac{(V_u - V_l)^2 \mu L}{h}}$$

5) $$\eta = \frac{q \Delta P}{\dot{W}} \Rightarrow \boxed{\eta = \frac{\frac{(V_l + V_u) h \Delta P}{2} - \frac{h^3 \Delta P^2}{12 \mu L}}{\frac{(V_l + V_u) h \Delta P}{2} + \frac{(V_u - V_l)^2 \mu L}{h}}} \rightarrow \boxed{\eta = \frac{1 - \frac{\Lambda}{6}}{1 + \left(\frac{V_u - V_l}{V_u + V_l}\right)^2 \frac{2}{\Lambda}}}$$

a) $V_l = V_u$: $$\eta = 1 - \frac{h^2 \Delta P}{6 \mu L (V_l + V_u)} \Rightarrow \boxed{\eta = 1 - \frac{\Lambda}{6}}$$



b) $V_l = 0$: $$\boxed{\eta = \frac{1 - \frac{\Lambda}{6}}{1 + \frac{2}{\Lambda}}}$$



Λ_{max}, η_{max} ACCOMPLISH THE CONDITION:

$$\frac{d\eta}{d\Lambda}\bigg|_{\Lambda_{max}} = 0 \Rightarrow \Lambda_{max}^2 + 4 \Lambda_{max} - 12 = 0 \Rightarrow \boxed{\Lambda_{max} = 2}$$

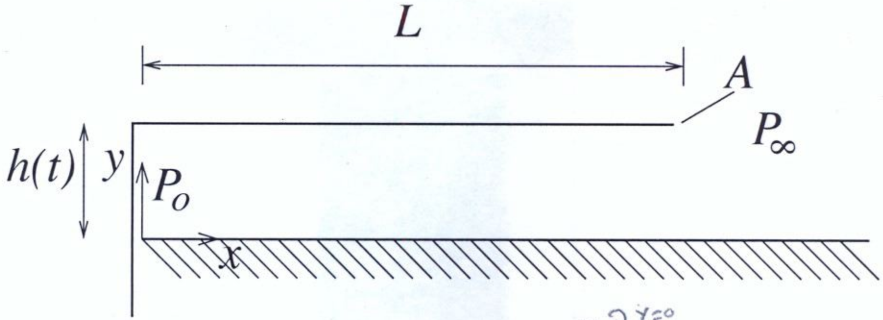
$$\boxed{\eta_{max} = \eta(\Lambda_{max}) = \frac{1}{3}}$$

INTRODUCTION TO FLUID MECHANICS

Jan 10, 2012

The L-shaped arm of length L depicted in the figure moves vertically approaching the horizontal wall, forming a thin channel of decreasing uniform thickness $h(t) \ll L$. A liquid of density ρ and viscosity μ fills the channel. Due to the arm displacement, the liquid is forced to move out, discharging to the surrounding atmosphere, where the reduced pressure is P_∞. For a known value of $h(t)$,

1. Give the conditions for the motion in the channel to be dominated by viscosity.
2. Determine the velocity profile across the channel $v_x(x, y, t)$, the associated volume flux $Q = \int_0^h v_x \mathrm{d}y$, and the reduced pressure distribution $P(x, t)$, including the value $P_o(t)$ at $x = 0$.
3. Calculate the force acting on the arm $\bar{F} = (F_x, F_y)$.
4. Obtain the torque exerted on the arm with respect to its end point A.



1) $\frac{\rho v_L h}{\mu}\frac{h}{L} \ll 1$, $v_L \sim \frac{L}{h}\dot{h} \Rightarrow \boxed{\frac{h\dot{h}}{\nu} \ll 1}$

$\frac{h^2/\nu}{t_o} \ll 1 \Rightarrow \boxed{\frac{h\dot{h}}{\nu} \ll 1}$

$t_o \simeq \frac{h}{\dot{h}}$

2) $0 = P_\ell + \mu \frac{\partial^2 v_x}{\partial y^2} \Rightarrow \boxed{v_x = \frac{P_\ell}{2\mu} y(h-y)}$

$v_x = 0$ @ $y = 0$, $y = h$

$Q = P_\ell \frac{h^3}{12\mu} = -\frac{\partial P}{\partial x}\frac{h^3}{12\mu}$

$\int_0^{h(t)} \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0\right) dy \Rightarrow \frac{\partial Q}{\partial x} + \frac{dh}{dt} = 0 \Rightarrow \boxed{-\frac{\partial}{\partial x}\left(P_\ell \frac{h^3}{12\mu}\right) + \frac{dh}{dt} = 0}$

$-Q = \frac{\partial P}{\partial x}\frac{h^3}{12\mu} = \frac{dh}{dt} x + \cancel{C_1}$ 0, BECAUSE $Q = 0$ AT $x = 0$

$\Rightarrow \boxed{P = \frac{12\mu}{h^3}\frac{dh}{dt}\frac{x^2}{2} + C_2 = \frac{6\mu}{h^3}\frac{dh}{dt}(x^2 - L^2) + P_\infty}$

$x = L$, $P = P_\infty$

$x = 0$, $P = \boxed{P_o = -\frac{6\mu}{h^3}\frac{dh}{dt}L^2 + P_\infty}$

3) $\boxed{F_y = \int_0^L (P - P_\infty)\, dx = -\frac{4\mu}{h^3}\frac{dh}{dt}L^3}$ $\boxed{F_x = -(P_o - P_\infty) h - \int_0^L \mu \left.\frac{\partial v_x}{\partial y}\right|_{y=h} dx = \frac{3\mu}{h^2}\frac{dh}{dt}L^2}$

$\frac{6\mu}{h^2}\frac{dh}{dt}L^2$ $\quad -\frac{3\mu}{h^2}\frac{dh}{dt}L^2$

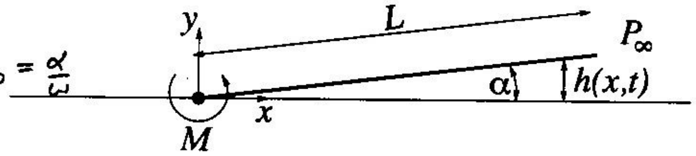
4) $\boxed{\bar{M} = -\int_0^L (P - P_\infty)(L - x)\, dx\; \bar{e}_z = -\int_0^L \frac{6\mu}{h^3}\frac{dh}{dt}(x^2 - L^2)(L - x)\, dx\; \bar{e}_z = +\frac{5}{2}\frac{\mu}{h^3}\frac{dh}{dt}L^4 \bar{e}_z}$

VF13 A plate of length L is initially sitting on a horizontal plane in the presence of a stagnant liquid atmosphere with reduced pressure P_∞. At a given instant, we begin to rotate the plate with constant angular velocity $\omega = d\alpha/dt$ by applying a given torque M at its left end, as sketched in the figure. For the analysis, use the approximation $h(x,t) = x\tan(\alpha) \simeq x\alpha$, valid for $\alpha \ll 1$.

1. Demonstrate that for values of α sufficiently smaller than a critical value, to be determined, the fluid motion in the gap formed between the plate and the wall is dominated by viscosity.
2. Obtain the velocity profile v_x in the gap as a function of the unknwon value of $P_l(x,t) = -\partial P/\partial x$ as well as the associated volume flux at a given section $Q = \int_0^{\alpha x} v_x \mathrm{d}y$.
3. Using continuity, write an equation linking Q and ω, and integrate it to compute the pressure distribution $P(x,t)$ (in the integration, you may anticipate that $x^3 P_l \to 0$ as $x \to 0$).
4. Determine the torque $M(t)$ needed to provide a constant angular velocity ω.

$h_c \sim L\alpha$

$\omega = \dfrac{d\alpha}{dt} \Rightarrow t_o = \dfrac{\alpha}{\omega}$



1) $\dfrac{h_c^2/(\mu/\rho)}{t_o} \sim \dfrac{L^2\alpha^2\omega}{\nu\alpha} \ll 1 \Rightarrow \boxed{\alpha \ll \dfrac{\nu}{L^2\omega}}$

$\dfrac{V_L h_c}{\nu}\dfrac{h_c}{L} \sim \dfrac{\omega L^2}{\nu}\dfrac{h_c}{L} \sim \dfrac{\omega L^2\alpha}{\nu} \ll 1$

$\dfrac{\partial V_x}{\partial x} + \dfrac{\partial V_y}{\partial y} = 0 \rightarrow V_L \sim \dfrac{\omega L^2}{h_c}$

2) $V_x = \dfrac{P_l}{2\mu} y(h-y) \rightarrow Q = \dfrac{P_l}{12\mu} h^3 = -\dfrac{x^3\alpha^3}{12\mu}\dfrac{\partial P}{\partial x}$

3) $\int_0^h \dfrac{\partial V_x}{\partial x}dy + \int_0^h \dfrac{\partial V_y}{\partial y}dy = 0 \Rightarrow \dfrac{\partial}{\partial x}Q - \cancel{\dfrac{dh}{dx}V_x(y=h)} + V_y(h) - \cancel{V_y(0)} = 0$

$\dfrac{\partial Q}{\partial x} + \omega x = 0$

$\dfrac{\partial}{\partial x}\left(\dfrac{\alpha^3}{12\mu}x^3\dfrac{\partial P}{\partial x}\right) - \omega x = 0 \Rightarrow \dfrac{\alpha^3}{12\mu}x^3\dfrac{\partial P}{\partial x} - \omega\dfrac{x^2}{2} = 0$

$\dfrac{\partial P}{\partial x} = \dfrac{6\mu\omega}{\alpha^3}\dfrac{1}{x} \Rightarrow \boxed{P - P_\infty = \dfrac{6\mu\omega}{\alpha^3}\ln\left(\dfrac{x}{L}\right)}$

4) $\bar{M}_{F\to Plate} = -\int_{\Sigma_P}(P-P_\infty)\bar{x}\wedge\bar{n}\,d\sigma = \int_0^L \dfrac{6\mu\omega}{\alpha^3}x\ln\dfrac{x}{L}\,dx\,\bar{e}_z = -\dfrac{3}{2}\dfrac{\mu\omega L^2}{\alpha^3}\bar{e}_z$

TO COUNTERBALANCE THIS MOMENTUM WE NEED TO APPLY A POSITIVE TORQUE $\boxed{\bar{M} = \dfrac{3}{2}\mu\dfrac{\omega L^2}{\alpha^3}\bar{e}_z}$

VF 12

The cylindrical bearing shown in the figure includes an external fixed casing of radius R_b and a shaft rotating with angular velocity ω of radius R such that $R_b - R \ll R$. The centers are shifted a small distance e, called exentricity, so that in between both cylinders there exists a thin circular gap of thickness $h \sim e \sim R_b - R \ll R$ full of a liquid of density ρ and viscosity μ.

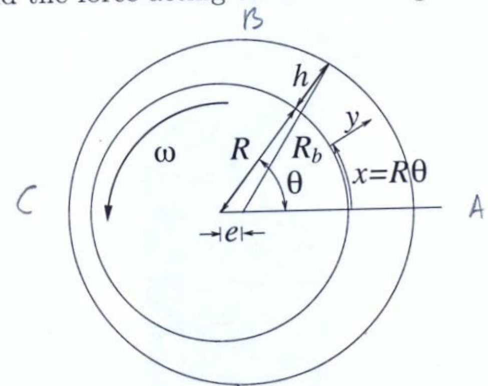
1. Show by simple trigonometric reasoning that the gap thickness can be expressed as

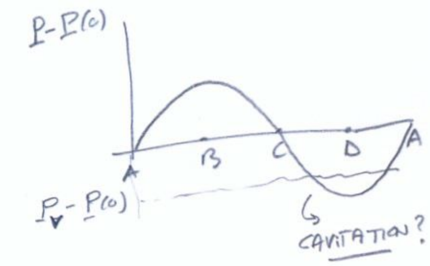
$$h(\theta) = R_b - R + e\cos\theta.$$

2. Obtain the azimuthal velocity distribution v_x in terms of the distance y to the fixed casing and the unknown value of the induced pressure gradient $P_l = -R^{-1}\partial P/\partial\theta$.

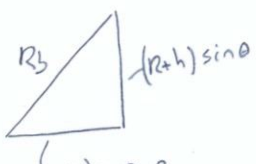
3. Determine the circulating volume flux $Q = \int_0^h v_x dy$ as well as the pressure distribution $P(\theta) - P(0)$, where $P(0)$ is the unknown value of the pressure at $\theta = 0$, which depends on the pressurization conditions.

4. Calculate the torque and the force acting on the rotating shaft.





1)



$R_b^2 = (R_b - R + R)^2 = (R+h)^2 - 2e(R+h)\cos\theta + e^2$

$\cancel{(R_b-R)^2} + \cancel{R^2} + 2(R_b - R)R = \cancel{R^2} + 2Rh + \cancel{h^2} - 2Re^{\cos\theta} - \cancel{2eh^{\cos\theta}} + \cancel{e^2}$

$e \sim h \sim R_b - R \ll R$ $\quad h = R_b - R + e\cos\theta = (R_b - R)[1 + \varepsilon\cos\theta]$, $\quad \varepsilon = \dfrac{e}{R_b - R}$

2) $v_x = \dfrac{\omega R(h-y)}{h} - \dfrac{y(h-y)}{2\mu}\dfrac{1}{R}\dfrac{\partial P}{\partial\theta} \rightarrow Q = \int_0^h v_x\,dy = \dfrac{\omega R h}{2} - \dfrac{h^3}{12\mu}\dfrac{1}{R}\dfrac{\partial P}{\partial\theta} \Rightarrow \dfrac{\partial P}{\partial\theta} = \dfrac{12\mu R}{h^3}\left(\dfrac{\omega R h}{2} - Q\right)$

INTEGRATING ONCE $\quad P(\theta) - P(0) = 12\mu R\left(\dfrac{\omega R}{2}\int_0^\theta \dfrac{d\theta}{h^2} - Q\int_0^\theta \dfrac{d\theta}{h^3}\right)$

SINCE $P(2\pi) = P(0) \Rightarrow Q = \dfrac{\omega R}{2}\dfrac{\int_0^{2\pi} d\theta/h^2}{\int_0^{2\pi} d\theta/h^3} = \dfrac{\omega R(R_b - R)}{2}\dfrac{\int_0^{2\pi}\frac{d\theta}{(1-\varepsilon\cos\theta)^2}}{\int_0^{2\pi}\frac{d\theta}{(1-\varepsilon\cos\theta)^3}} = \omega R(R_b - R)\dfrac{1-\varepsilon^2}{2+\varepsilon^2}$

$P(\theta) - P(0) = \dfrac{6\mu\omega R^2}{(R_b - R)^2}\dfrac{\varepsilon(2+\varepsilon\cos\theta)\sin\theta}{(2+\varepsilon^2)(1+\varepsilon\cos\theta)^2}$ $\quad P(\theta) = -P(-\theta) \Rightarrow$ VERTICAL FORCE

P_{min} AT $\theta = \frac{3\pi}{2} \Rightarrow$ CAVITATION!

FORCE ON THE SHAFT $\quad F_z = -\int_0^{2\pi}(P - P(0))\, n_z\, R\,d\theta = -R\int_0^{2\pi}(P - P(0))\sin\theta\, d\theta = -\dfrac{12\mu\omega R^3\pi}{(R_b - R)^2}\dfrac{\varepsilon}{(1-\varepsilon^2)^{1/2}(2+\varepsilon^2)}$ $\Downarrow$

FORCE ON THE BEARING $\quad + \dfrac{12\mu\omega R^3\pi}{(R_b - R)^2}\dfrac{\varepsilon}{(1-\varepsilon^2)^{1/2}(2+\varepsilon^2)}$ $\Uparrow$

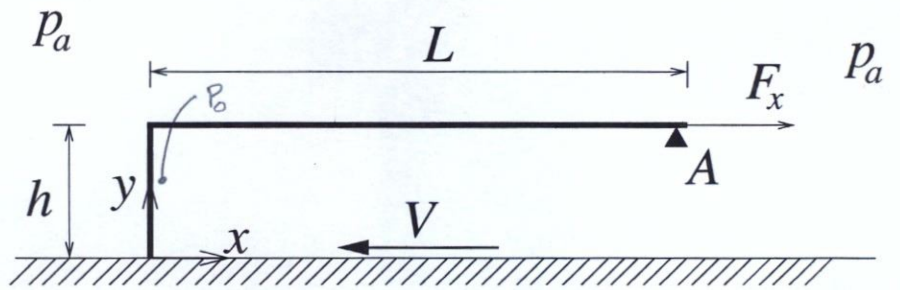
TORQUE $\quad \tau_p = \mu\left.\dfrac{\partial v_x}{\partial y}\right|_{y=0} = -\mu\dfrac{\omega R}{h} - \dfrac{h}{2R}\dfrac{\partial P}{\partial\theta}$, $\quad M = \int_0^{2\pi}\tau_p R\,R\,d\theta = -\dfrac{4\pi\mu\omega R^3}{R_b - R}\dfrac{(1+2\varepsilon^2)}{(1-\varepsilon^2)^{1/2}(2+\varepsilon^2)}$

BEARINGS CAN IN PRINCIPLE WITHSTAND AN INFINITE FORCE BECAUSE AS $F_z \to \infty$, $\varepsilon \to 1$.
HOWEVER, FAILURE APPEARS WHEN THE MINIMUM PRESSURE REACHES P_v AND THE LIQUID CAVITATES.

N=11

To investigate the performance of windshield wipers consider as a simplified model the L-shaped arm of the figure moving with velocity V with respect to a horizontal surface. The wiper is pulled from the end point A with a force F_x parallel to the wall, dragging in its motion the fluid located between the arm and the wall. To study the problem, we shall use a reference frame moving with the arm, as indicated in the figure. The outer pressure is p_a.

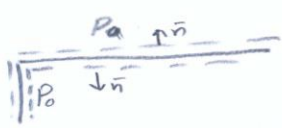
1. Determine the velocity v_x across the channel formed between the wiper and the wall.
2. Find the pressure distribution $p(x)$ including the value found at the end wall $x = 0$.
3. Compute the force F_x necessary to move the wiper.
4. Obtain the moment exerted with respect to the end point A.
5. If W is the weight of the wiper, determine the critical value of V beyond which the end $x = 0$ rises, letting the liquid flow out.



1) $V_x = -V\frac{h-y}{h} + \frac{P_\ell}{2\mu} h^2 \frac{y}{h}\left(1-\frac{y}{h}\right)$, 2) $Q = \int_0^h V_x\, dy = \frac{P_\ell h^3}{12\mu} - \frac{Vh}{2} = 0 \Rightarrow P_\ell = \frac{6\mu V}{h^2} = \frac{P_0 - P_a}{L}$

$\boxed{P_0 = P_a + \frac{6\mu V L}{h^2}}$

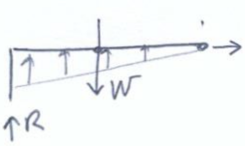
$\boxed{P - P_a = \frac{6\mu V}{h^2}(L-x)}$

3) 

$\left(-\int_\Sigma (P-P_a)\,\bar n\, d\sigma + \int_\Sigma \bar{\bar\tau}'\cdot\bar n\, d\sigma\right)_x = -\overbrace{\frac{6\mu V L}{h^2}}^{P_0-P_a} h\,\bar e_x - \mu\left(\frac{dv_x}{dy}\right)_{y=h} L\,\bar e_x = -4\frac{\mu V L}{h}\bar e_x$

$\boxed{F_x = 4\mu\frac{VL}{h}}$

4) $\bar M_{F\to A} = -\int(\bar x-\bar x_0)\wedge(P-P_a)\,\bar n\, d\sigma + \cancel{\int(\bar x-\bar x_0)\wedge\bar{\bar\tau}'\cdot\bar n\, d\sigma} = -\int_0^L (P-P_a)(L-x)\,dx\,\bar e_z = -2\frac{\mu V L^3}{h^2}\bar e_z$

5) 

$\bar M_A = \bar M_{F\to A} + W\frac{L}{2}\bar e_z - RL\,\bar e_z \rightarrow$ THE WIPER BEGIN TO RISED WHEN $R=0$ $\boxed{V = \frac{W h^2}{4\mu L^2}}$

VF10 Consider the unidirectional periodic motion induced in an infinitely long circular pipe of radius a by an oscillatory pressure gradient $P_l = A\cos(\omega t)$. Write the conservation equation with boundary conditions that determine the axial velocity $v_x(r,t)$ and show how the problem can be solved exactly by separation of variables. Study separately the limits $a^2\omega/\nu \gg 1$ and $a^2\omega/\nu \ll 1$ and obtain the corresponding limiting solutions. In biofluid mechanics the square root $a(\omega/\nu)^{1/2}$ is called the Womersley parameter, which takes fairly large values for blood flow in large arteries, as you can see by using the values corresponding to the human aorta ($\mu/\rho \simeq 4\times 10^{-2}$ cm^2/s, $a \simeq 1.2$ cm, and $\omega = 2\pi$ s^{-1}), but that decreases for flow is smaller arteries. Find how small the artery radius needs to be for Poiseuille flow to be approximately applicable.

$$\underset{\rho V_L/\omega^{-1}}{\rho \frac{\partial V_x}{\partial t}} = \underset{A}{A\cos(\omega t)} + \underset{\mu V_L/a^2}{\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_x}{\partial r}\right)} \quad \begin{cases} r=0: \dfrac{\partial V_x}{\partial r}=0 \\ r=a: V_x=0 \end{cases}$$

EXACT SOLUTION:

$$\rho\frac{\partial V^*}{\partial t} = Ae^{i\omega t} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V^*}{\partial r}\right) \quad \begin{cases} r=0: \dfrac{\partial V^*}{\partial r}=0 \\ r=a: V^*=0 \end{cases}$$

$$V^* = Ae^{i\omega t}f(r) \rightarrow \quad \rho i\omega f = 1 + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) \quad \begin{cases} \dfrac{\partial f}{\partial r}=0 \text{ AT } r=0 \\ f=0 \text{ AT } r=a \end{cases}$$

$$\frac{O\left(\rho\frac{\partial V_x}{\partial t}\right)}{O\left(\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_x}{\partial r}\right)\right)} \sim a^2\frac{\omega}{\nu}$$

$$a^2\frac{\omega}{\nu} \ll 1 \Rightarrow 0 = A\cos(\omega t) + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_x}{\partial r}\right) \Rightarrow \boxed{V_x = \frac{A\cos(\omega t)}{4\mu}(a^2-r^2)}$$

QUASI-STEADY POISEUILLE FLOW

$$a^2\frac{\omega}{\nu} \gg 1 \Rightarrow \rho\frac{\partial V_x}{\partial t} = A\cos(\omega t) \Rightarrow \boxed{V_x = \frac{A}{\rho\omega}\sin(\omega t)}$$

UNIFORM FLOW

THE SOLUTION DOES NOT SATISFY THE BOUNDARY CONDITION ON THE WALL. IN ITS VICINITY, THERE EXISTS A THIN BOUNDARY LAYER WHERE VISCOSITY IS IMPORTANT $\rho\frac{V_L}{\omega^{-1}} \sim \mu\frac{V_L}{\delta^2} \Rightarrow \delta \sim \sqrt{\nu/\omega}$

INTRODUCING $\eta = \dfrac{r-y}{\sqrt{\nu/\omega}}$, $\dfrac{A}{\rho\omega}\sin(\omega t) - V_x = \tilde{V}$, $\omega t = \tau$

$$\frac{\partial\tilde{V}}{\partial\tau} = \frac{\partial^2\tilde{V}}{\partial\eta^2} \quad \begin{cases} \eta=0: \tilde{V} = \dfrac{A}{\rho\omega}\sin(\omega t) \\ \eta\to\infty: \tilde{V}\to 0 \end{cases} \quad \text{STOKES SOLUTION!!}$$

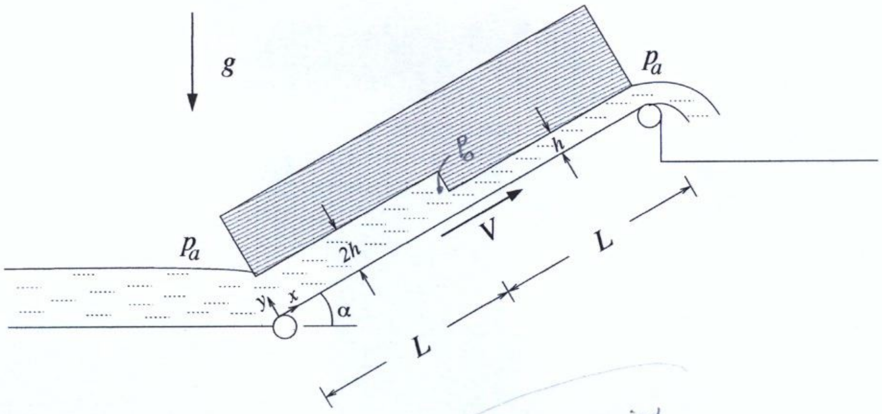
BLOOD FLOW IN THE HUMAN AORTA $\omega \simeq 2\pi$ s^{-1}, $\nu \sim 4\times10^{-2}$ cm^2/s, $a \simeq 1.2$cm

$a^2\dfrac{\omega}{\nu} \simeq 226$, Womersley $\simeq 15$

VISCOSITY BECOMES IMPORTANT FOR $a \sim \sqrt{\dfrac{\nu}{\omega}} \sim 0.8$ mm AND IS DOMINANT FOR $a \ll 0.8$ mm

VF9 The conveyor belt of velocity V shown in the figure is used to pump a liquid of density ρ and viscosity μ along a straight inclined channel of angle α including two stretches of equal length L with different heights $2h$ and h, respectively.

- Give the conditions for the liquid-film motion to be dominated by viscosity.
- Compute the pressure distribution along the channel $p(x)$.
- Determine the volume flux Q as well as the minimum value of V for which $Q > 0$.
- Obtain the power needed to drive the conveyor belt.



VISCOSITY DOMINATES IF $\frac{\rho V h}{\mu}\frac{h}{L} \ll 1$

$$P_{\ell_1} = \frac{P_a - P_0 - \rho g L \sin\alpha}{L}$$

$0<x<L$, $\quad V_x = +\frac{2h^2}{\mu} P_{\ell_1} \frac{y}{2h}\left(1-\frac{y}{2h}\right) + V\frac{2h-y}{2h} \rightarrow \boxed{Q = \int_0^{2h} V_x\,dy = \frac{2}{3}\frac{h^3}{\mu}\frac{P_a - P_0 - \rho g L\sin\alpha}{L} + Vh}$

$L<x<2L$, $\quad V_x = \frac{h^2 P_{\ell_2}}{2\mu}\frac{y}{h}\left(1-\frac{y}{h}\right) + V\frac{h-y}{h} \rightarrow \boxed{Q = \frac{h^3}{12\mu}\frac{P_0 - P_a - \rho g L\sin\alpha}{L} + \frac{Vh}{2}}$

$$P_{\ell_2} = P_0 - P_a - \rho g L \sin\alpha$$

FROM THE TWO EQUATIONS $\boxed{P_0 - P_a = \frac{2}{3}\frac{\mu V L}{h^2} - \frac{7}{9}\rho g L\sin\alpha}$ AND $\boxed{Q = \frac{5}{9}Vh - \frac{4}{27}\rho g\frac{h^3\sin\alpha}{\mu}}$

$$\hookrightarrow V_{min} = \frac{4}{15}\frac{\rho g h^2 \sin\alpha}{\mu}$$

$0<x<L$:
$$-\frac{\partial(P)}{\partial x} = \frac{P_a - (P_0 + \rho g L\sin\alpha)}{L} \Rightarrow P + \rho g x\sin\alpha = P_a - \frac{P_a - (P_0 + \rho g L\sin\alpha)}{L}x \rightarrow \boxed{P = P_a + (P_0 - P_a)\frac{x}{L}}$$

$L<x<2L$: $$-\frac{\partial P}{\partial x} = \frac{P_0 + \rho g L\sin\alpha - (P_a + 2\rho g L\sin\alpha)}{L} \Rightarrow \boxed{P = P_0 - (P_0 - P_a)\frac{x-L}{L}}$$

$0<x<L$: $\tau_1 = \mu\left.\frac{\partial V_x}{\partial y}\right)_{y=0} = -\frac{\mu V}{2h} + P_{\ell_1} h$

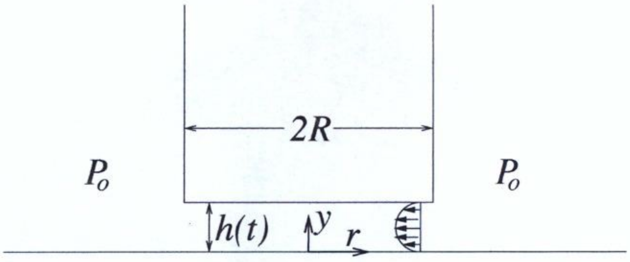
$L<x<2L$: $\tau_2 = \mu\left.\frac{\partial V_x}{\partial y}\right)_{y=0} = -\frac{\mu V}{h} + \frac{P_{\ell_2} h}{2}$

$$W = VL(\tau_1 + \tau_2) = VL\left(-\frac{3}{2}\frac{\mu V}{h} - \frac{3}{2}\rho g\sin\alpha\, h - \frac{1}{2}\frac{P_0 - P_a}{L}h\right)$$

VF8

We want to lift the cylindrical body shown in the figure. A layer of oil exists between the body and the horizontal solid surface. The radial inflow caused by the lifting motion of the body is associated with the existence of underpressures in the liquid film, inducing an adhesion force to be computed here for the case of constant velocity $\dot{h} = dh/dt$.

- Show that the liquid motion in the liquid film between the body and the solid surface is dominated by viscosity for values of h much smaller than $\nu/\dot{h}$.
- Obtain the radial velocity profile v_r as a function of the radial pressure gradiente $P_l = -\partial P/\partial r$.
- Using continuity, derive an equation for the pressure distribution $P(r,t)$.
- Compute the pressuse distribution as well as the adhesion force acting on the body.



1) $\frac{\partial r v_r}{\partial r} + \frac{\partial r v_y}{\partial y} = 0$

$v_L \sim \frac{R\dot{h}}{h}$

$t_o \sim \frac{h}{\dot{h}}$

$$\left.\begin{array}{l} \frac{O(\rho \partial v/\partial t)}{O(\nabla\cdot\bar{\bar{\tau}}')} = \frac{h^2/\nu}{t_o} \sim \frac{h\dot{h}}{\nu} \ll 1 \\ \frac{O(\rho\bar{v}\cdot\nabla\bar{v})}{O(\nabla\cdot\bar{\bar{\tau}}')} = \frac{v_L h}{\nu}\frac{h}{R} \sim \frac{h\dot{h}}{\nu} \ll 1 \end{array}\right\} h \ll \nu/\dot{h}$$

2) $$0 = -\frac{\partial P}{\partial r} + \mu\frac{\partial^2 v_r}{\partial y^2} \Rightarrow v_r = -\frac{h^2}{2\mu}\frac{\partial P}{\partial r}\frac{y}{h}\left(1-\frac{y}{h}\right)$$

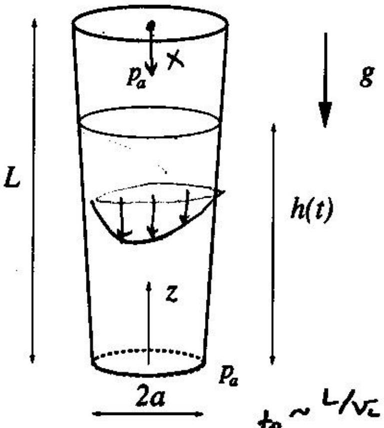
3) FROM CONTINUITY $$\int_0^h\left(\frac{\partial r v_r}{\partial r} + \frac{\partial r v_y}{\partial y}\right)dy = \frac{\partial}{\partial r}\int_0^h r v_r\, dy + r\left[\overbrace{v_y(h)}^{\dot{h}} - v_y(0)\right] = 0$$

4) $$-\frac{\partial}{\partial r}\left(r\frac{h^3}{12\mu}\frac{\partial P}{\partial r}\right) + r\dot{h} = 0 \quad \left\{\begin{array}{l} r=0:\ \frac{\partial P}{\partial r}=0 \\ r=R,\ P=P_o \end{array}\right. \Rightarrow \boxed{P - P_o = \frac{3\mu}{h^3}\dot{h}(r^2-R^2)}$$

$$\boxed{F = \int_0^R 2\pi r(P-P_o)\,dr\,\bar{e}_y = -\frac{3\pi}{2}\mu\frac{R^4}{h^3}\dot{h}}$$

VF7

The vertical conical pipe of length L shown in the figure has a radius that increases linearly from the lower end according to $a(z) = a_o(1+\alpha z/L)$, with $a_o \ll 1$ and $\alpha \sim O(1)$. The pipe, open at the top, is initially full of a liquid of viscosity μ and density ρ. At a given time, the lower end opens up and the liquid discharges to the atmosphere under the action of gravity. Give the condition for the motion to be dominated by viscosity and, in that case, obtain the evolution of $h(t)$ as well as the time needed to complete the discharge.



IF VISCOSITY DOMINATES $\dfrac{\Delta_L P}{L} \sim \mu \dfrac{V_L}{a_o^2} \rightarrow V_L \sim \dfrac{a_o^2 \Delta_L P}{\mu L} \sim \dfrac{\rho g a_o^2}{\mu}$

$t_o \sim L/V_L$

THE MOTION IS DOMINATED BY VISCOSITY IF $\dfrac{a_o^2/(\mu/\rho)}{t_o} \sim \rho \dfrac{V_L}{\mu} \dfrac{a_o a_o}{L} \sim \boxed{\dfrac{\rho^2 g a_o^4}{\mu^2 L} \ll 1}$

$$0 = -\frac{\partial P}{\partial x} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right) \rightarrow v_x = \frac{P_\ell}{4\mu}(a^2-r^2), \quad Q(t) = \frac{\pi}{8\mu}a^4 P_\ell$$

FROM CONTINUITY

$$\frac{\partial}{\partial x}(r v_x) + \frac{\partial}{\partial r}(r v_r) = 0$$

$$\frac{\partial}{\partial x}\int_0^a r v_x \, dr + r\left[v_r(a) - v_r(0)\right] = 0$$

$$-\frac{da}{dx} r v_x|_a$$

$$\frac{\partial}{\partial x}\int_0^a 2\pi r v_x \, dr = 0 \rightarrow Q(t)$$

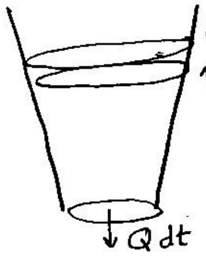
$x = L-h$, $P = P_a + \rho g h$

$x = L$, $P = P_a$

$$-\frac{\partial P}{\partial x} = -\frac{8\mu Q}{\pi a^4} \Rightarrow P_a + \rho g h - P = \frac{8\mu Q}{\pi}\int_{L-h}^{x}\frac{dx}{a^4} \Rightarrow \rho g h = \frac{8\mu Q}{\pi a_o^4}\int_{L-h}^{L}\frac{dx}{\left[1+\frac{\alpha}{L}(L-x)\right]}$$

$$\boxed{\rho g h = \frac{8\mu Q L}{3\alpha\pi a_o^4}\left[1 - \frac{1}{\left(1+\alpha\frac{h}{L}\right)^3}\right]}$$

TO CLOSE THE PROBLEM WE USE CONTINUITY TO WRITE



$dV = -\pi a^2(h)\, dh$

$$\boxed{Q = -\pi a^2(h)\frac{dh}{dt}}$$

$$\rho g h = -\frac{8}{3\cancel{\pi}\alpha}\frac{\mu L}{a_o^{\cancel{4}2}}\left[1 - \frac{1}{\left(1+\alpha\frac{h}{L}\right)^3}\right]\cancel{\pi}\cancel{a_o^2}\left(1+\alpha\frac{h}{L}\right)^2\frac{dh}{dt}$$

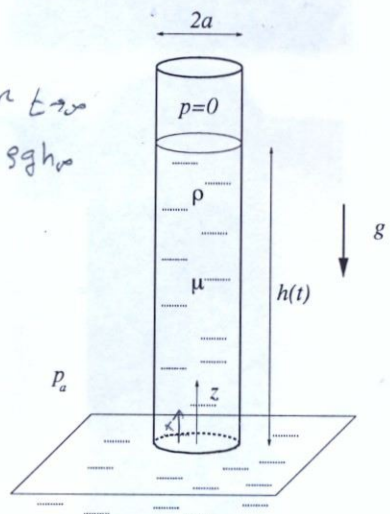
$$\boxed{\frac{3\alpha}{8}\frac{\rho g a_o^2}{\mu L}\int_0^t dt = -\int_L^h \frac{\left(1+\frac{\alpha h}{L}\right)^3 - 1}{h\left(1+\frac{\alpha h}{L}\right)}\, dh}$$

$h = 0$ WHEN

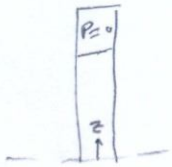
$$\boxed{t_d = \frac{8}{3\alpha}\frac{\mu L}{\rho g a_o^2}\int_0^L \frac{\left(1+\frac{\alpha h}{L}\right)^3 - 1}{h\left(1+\frac{\alpha h}{L}\right)}\, dh}$$

VF6 The bottom end of an empty vertical tube of radius a, closed at the top, is put in contact with a pool of oil. Because of the ambient overpressure, the liquid begins to flow into the tube, forming a column of increasing height $h(t)$ whose evolution in time is to be investigated assuming that the motion is dominated by viscosity. In particular:

- Obtain the value of the height h_∞ corresponding to the equilibrium position, reached asymptotically for large times.
- Give the condition that determines whether the motion is dominated by viscosity.
- Obtain the evolution of $h(t)$ as well as the pressure distribution along the pipe $p(x,t)$ for $0 < x < h$.
- Compute the force acting on the pipe as a function of time $\bar{F} = F_z(t)\bar{e}_z$.



1) IN THE LIQUID IN EQUILIBRIUM FOR $t \to \infty$



$$P + \rho g z = \text{CONSTANT} = P_a = \rho g h_\infty$$

$$\boxed{h_\infty = \frac{P_a}{\rho g}}$$

$x \equiv z$

2) IF DOMINATED BY VISCOSITY, $V_L \sim \frac{D^2}{L}\frac{P_a}{\mu}$

$$\frac{O(\rho \frac{\partial v_x}{\partial t})}{O(\mu \nabla^2 \bar{v})} \sim \frac{D^2/(\mu/\rho)}{t_o} \sim \frac{D^2/(\mu/\rho)}{L/V_L} \sim \frac{\rho V_L D}{\mu}\frac{D}{L}$$

$$\frac{O(\rho \bar{v}\cdot\nabla\bar{v})}{O(\mu\nabla^2\bar{v})} \sim \frac{\rho V_L D}{\mu}\frac{D}{L}$$

CRITERION $\frac{\rho V_L D}{\mu}\frac{D}{L} \sim \frac{\rho D^4 P_a}{\mu^2 h_\infty^2} \sim \frac{\rho^3 g^2 D^4}{\mu^2 P_a} \ll 1$

3) $$0 = \underbrace{-\frac{\partial P}{\partial x}}_{P_\ell} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right) \xrightarrow{r=a,\ v_x=0} V_x = \frac{P_\ell}{4\mu}(a^2 - r^2) \rightarrow Q = \int_0^a 2\pi v_x r\,dr = \frac{\pi}{8\mu}P_\ell a^4$$

$$P_\ell(t) = -\frac{\partial P}{\partial x} = +\frac{8\mu Q(t)}{\pi a^2} \Rightarrow P = p + \rho g z = -\int \frac{8\mu Q}{\pi a^2}dx + C_1 = C_1 - \frac{8\mu Q}{\pi a^2}x$$

$x = 0$, $P = P_a \rightarrow C_1 = P_a$

$$\boxed{P + \rho g z = P_a - \frac{8\mu Q}{\pi a^2}z}$$

$x = h$, $P = \rho g h$

$$\boxed{P_a - \rho g h = \frac{8\mu Q}{\pi a^2}h}$$

$$Q = \cancel{\pi a^2}\frac{dh}{dt} = \frac{P_a - \rho g h}{h}\frac{\cancel{\pi a^2}}{8\mu}$$

$$\frac{dh}{dt} = \frac{P_a}{8\mu}\frac{1 - h/h_\infty}{h}$$

$$\int_0^h \frac{h\,dh}{1 - h/h_\infty} = \int_0^t (P_a/8\mu)\,dt$$

$$\boxed{\frac{P_a}{8\mu h_\infty^2}t = \ln\left(\frac{1}{1 - h/h_\infty}\right) - \frac{h}{h_\infty}}$$

$\hookrightarrow h(t)$ | AS $t \to \infty$, $h \to h_\infty$

4)

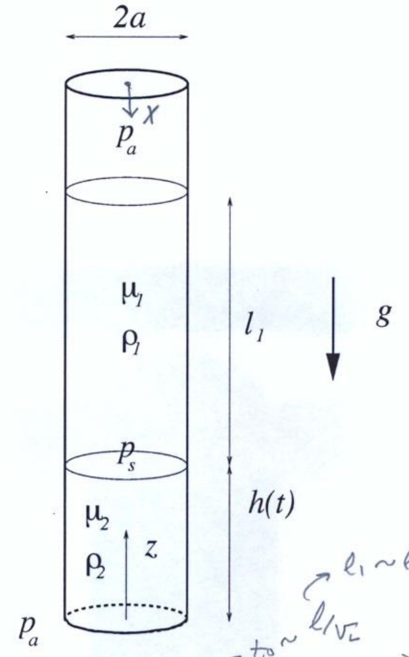
$$\tau_p = -\mu\frac{\partial v_x}{\partial r} = \frac{P_\ell a}{2} = \frac{a}{2}\frac{P_a - \rho g h}{h}$$

$$F_p = \cancel{2}\pi a h \frac{a}{\cancel{2}}\frac{P_a - \rho g h}{h} = \pi a^2 (P_a - \rho g h)$$

THE TOTAL FORCE IS $\underbrace{\pi a^2 (P_a - \rho g h)\bar{e}_z}_{\text{VISCOUS FORCE ON THE WALL}} - \underbrace{P_a \pi a^2}_{\text{PRESSURE FORCE ON THE TOP END}} = -\rho g h \pi a^2 \bar{e}_z$

VFS

A vertical tube of radius a open at the top contains initially two columns of heights l_1 and l_2 of two inmiscible liquids with different densities ρ_1 and ρ_2 and viscosities μ_1 and μ_2. At a given instant of time, the bottom end of the pipe opens up, releasing the fluid 2 to the atmosphere. During the discharge process, determine the volume flux as a function of time as well as the pressure distribution along the pipe. Calculate the time for complete discharge of the bottom liquid column.



$t_0 \sim l/V_c$, $l_1 \sim l_2$, $V_c \sim \frac{\Delta P\, a^2}{\mu l} \sim \frac{\rho g a^2}{\mu}$

THE FLOW IS DOMINATED BY VISCOSITY IF $\frac{a^2/(\mu/\rho)}{t_0} \sim \frac{\rho V_c a}{\mu}\frac{a}{l} \sim \frac{\rho^2 g a^4}{\mu^2 l} \ll 1$

TOP FLUID $\quad V_x = \frac{P_{l_1}}{4\mu_1}(a^2-r^2), \quad Q_1 = \frac{\pi}{8\mu_1}a^4 P_{l_1} = \frac{\pi}{8\mu_1}a^4\frac{(P_a-P_s+\rho_1 g l_1)}{l_1}$

BOTTOM FLUID $\quad Q_2 = \frac{\pi}{8\mu_2}a^4 P_{l_2} = \frac{\pi}{8\mu_2}a^4\frac{P_s-P_a+\rho_2 g h}{h}$

$dV = -\pi a^2 dh$

$Q_1 = Q_2 = Q \Rightarrow$

$$P_a - P_s + \rho_1 g l_1 = \frac{8}{\pi}\frac{\mu_1 l_1}{a^4}Q$$

$$P_s - P_a + \rho_2 g h = \frac{8}{\pi}\frac{\mu_2 h}{a^4}Q$$

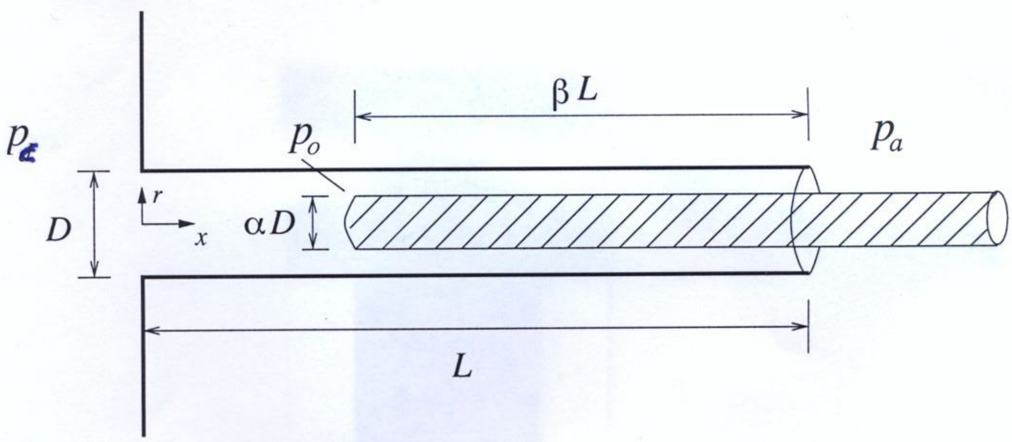
$$Q = -\pi a^2\frac{dh}{dt}$$

$$(\rho_1 l_1 + \rho_2 h)g = \frac{8}{\pi a^4}(\mu_1 l_1 + \mu_2 h)Q = -\frac{8}{a^2}(\mu_1 l_1+\mu_2 h)\frac{dh}{dt}$$

$$\boxed{\int_{l_2}^{h}\frac{(\mu_1 l_1+\mu_2 h)\,dh}{(\rho_1 l_1+\rho_2 h)} = -\frac{g a^2}{8}\int_0^t dt}$$

VF4 The liquid ~~contained in a deposit~~ LOCATED IN A CONTAINER at pressure p_c discharges to the atmosphere, where the pressure is $p_a < p_c$, through a cylindrical pipe of diameter D and length $L \gg D$. To control the volume flux Q that circulates, a cylindrical bar of diameter αD is inserted coaxially from the exit, as shown in the figure, to partially block the liquid flow in the final stretch of the pipe.

1. Give the criterion for the motion in the tube to be dominated by viscosity.
2. Determine, in terms of the value of the pressure at the end of the tube p_o, the velocity field found for $0 < x/L < (1-\beta)$ and for $(1-\beta) < x/L < L$, where βL is the insertion length.
3. Obtain the volume flux Q and the value of p_o as a function of β.



1) THE MOTION IS STEADY. CONVECTION IS NEGLIGIBLE IF $\frac{\rho V_L D}{\mu}\frac{D}{L} \ll 1$, TO ESTIMATE V_L WE ASSUME THAT VISCOSITY IN FACT DOMINATES, SO THAT $\frac{\Delta_L P}{L} \sim \mu \frac{V_T}{D^2} \rightarrow V_T \sim \frac{D^2}{L}\frac{\Delta_L P}{\mu} \sim \frac{D^2}{L}\frac{P_c - P_a}{\mu}$

IN TERMS OF KNOWN QUANTITIES, THE CRITERION TO BE SATISFIED IS

$$\boxed{\frac{\rho D^4}{\mu^2 L^2}(P_c - P_a) \ll 1}$$

2) IF THE ABOVE CONDITION HOLDS, THEN WE FIND DEVELOPED FLOW EVERYWHERE ALONG THE PIPE, EXCEPT IN SMALL REGIONS NEAR THE ENTRANCE AND ALSO NEAR THE TRANSITION REGION $x = L(1-\beta)$, BUT THE ASSOCIATED PRESSURE VARIATIONS ARE SMALL, SO WE CAN NEGLECT THEM IN THE FIRST APPROXIMATION AND USE $P(0) = P_d$ TOGETHER WITH A VALUE $P = P_o$ AT THE END OF THE FIRST STRETCH EQUAL TO THAT FOUND AT THE ENTRANCE OF THE SECOND STRETCH.

$$0 < x < (1-\beta)L \rightarrow V_x = \frac{1}{4\mu}\frac{dP}{dx}\left[r^2 - \left(\frac{D}{2}\right)^2\right] \Rightarrow \boxed{Q = \frac{P_c - P_o}{(1-\beta)L}\frac{\pi D^4}{128\mu}} \quad (1)$$

$$0 = P_\ell + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right)$$

UNIDIRECTIONAL MOTION WITH D = CONSTANT

SINCE $V_x(r) \rightarrow \boxed{P_\ell = \text{CONSTANT}} \rightarrow \frac{dP}{dx} = \text{CONSTANT} = \frac{P_c - P_o}{(1-\beta)L}$

$$1-\beta < \frac{x}{L} < 1 \rightarrow V_x = -\frac{dP}{dx}\frac{D^2}{16\mu}\left[\left(1-\left(\frac{2r}{D}\right)^2\right) - \frac{1-\alpha^2}{\ln\alpha}\ln\left(\frac{2r}{D}\right)\right] \Rightarrow \boxed{Q = \int_{\alpha\frac{D}{2}}^{D/2} 2\pi r V_x dx = \frac{P_o - P_a}{\beta L}\frac{\pi D^4}{128\mu}f(\alpha)} \quad (2)$$

$$-\frac{dP}{dx} = \frac{P_o - P_a}{\beta L}$$

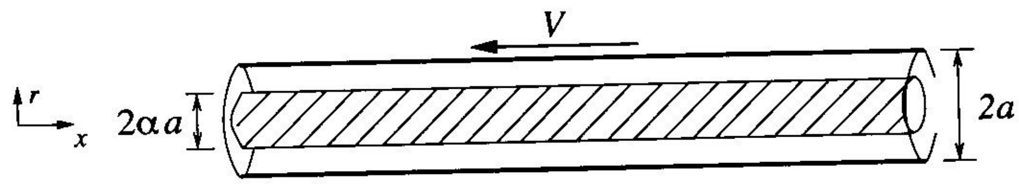
$$f(\alpha) = 1 - \alpha^2(2-\alpha^2) - \frac{(1-\alpha^2)}{\ln(\alpha^{-1})}\left[1 - \alpha^2 - 2\alpha^2\ln(\alpha^{-1})\right]$$

EQUATING (1) AND (2) GIVES

$$\boxed{Q = \frac{\pi D^4}{128\mu L}\frac{P_c - P_a}{1 + \frac{\beta f}{1-f}}}, \quad \boxed{P_o = P_c - \frac{(1-\beta)(P_c - P_a)}{1 + \beta f/(1-f)}}$$

VF3 A fluid of density ρ and viscosity μ is confined between two infinitely long coaxial cylinders of radii a and αa, with $\alpha < 1$, aligned in the x direction. The fluid is forced to move steadily due to the presence of a known reduced pressure gradient $P_l = -\partial P/\partial x$ and also due to the backward motion of the outer cylinder, with velocity V.

1. Determine the fluid velocity $v_x(r)$.
2. Obtain the power needed to move the outer cylinder.
3. Calculate the value of V for which the volume flux in the pipe is identically zero.



$$0 = P_l + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right) \quad \left\{ \begin{array}{l} r=a: \ v_x = -V \\ r=\alpha a: \ v_x = 0 \end{array} \right\}$$

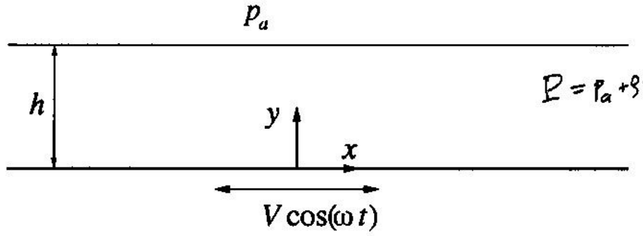
INTEGRATING TWICE $\rightarrow$ $\Big[\, v_x = -\frac{P_l}{\mu}\frac{r^2}{4} + C_1 \ln r + C_2 = \boxed{-\frac{P_l}{4\mu}\left[r^2 - (\alpha a)^2\right] + \left[-V + \frac{P_l}{\mu}\frac{(1-\alpha^2)a^2}{4}\right]\frac{\ln\left(\frac{r}{\alpha a}\right)}{\ln(\alpha^{-1})}}$

$$\tau_p = -\mu \left.\frac{\partial v_x}{\partial r}\right)_{r=a} = \frac{P_l a}{2}\left[1 - \frac{1-\alpha^2}{2\ln(\alpha^{-1})}\right] + \frac{\mu V}{a\ln(\alpha^{-1})}, \quad \boxed{W = \tau_p V 2\pi a}$$

$$Q = \int_{\alpha a}^{a} 2\pi r v_x \, dr = 0 \qquad \boxed{\frac{V}{\ln(\alpha^{-1})}\int_{\alpha a}^{a} r\ln(r/\alpha a)\,dr = \frac{P_l}{4\mu}\int_{\alpha a}^{a}\left[(1-\alpha^2)a^2\frac{\ln(r/\alpha a)}{\ln(\alpha^{-1})} - r^2 + (\alpha a)^2\right] r\,dr}$$

VF2

Consider the periodic motion induced in a liquid layer of thickness h when the bottom wall moves with velocity $V\cos(\omega t)$. Write the equation with boundary conditions that determines $v_x(y,t)$. Consider separately the cases $\omega h^2/\nu \gg 1$ and $\omega h^2/\nu \ll 1$. Obtain the general solution and show how it reduces to the limiting solutions studied earlier in the appriate limits.



$$P = p_a + \rho g h \Rightarrow \frac{\partial P}{\partial x} = 0$$

$$\frac{\partial V_x}{\partial t} = \nu \frac{\partial^2 V_x}{\partial y^2} \quad \begin{cases} y=0: & V_x = V\cos(\omega t) \\ y=h: & \dfrac{\partial V_x}{\partial y} = 0 \end{cases}$$ (NEGLECTING VISCOSITY OF AIR)

EXACT SOLUTION:

$$\frac{\partial V^*}{\partial t} = \nu \frac{\partial^2 V^*}{\partial y^2} \quad \begin{cases} y=0: & V^* = V e^{i\omega t} \\ y=h: & \dfrac{\partial V^*}{\partial y} = 0 \end{cases} \rightarrow V^* = V e^{i\omega t} f(y)$$

$$i\frac{\omega}{\nu} f = f'' \qquad f(0)=1, \quad f'(h)=0$$

$$f = C_1 e^{(1+i)\sqrt{\frac{\omega}{2\nu}}\,y} + C_2 e^{-(1+i)\sqrt{\frac{\omega}{2\nu}}\,y}$$

$$\boxed{\begin{aligned} & C_1 + C_2 = 1 \\ & C_1 e^{(1+i)\sqrt{\frac{\omega}{2\nu}}\,h} - C_2 e^{-(1+i)\sqrt{\frac{\omega}{2\nu}}\,h} = 1 \end{aligned}}$$

$$\boxed{C_1 = \frac{e^{-\sqrt{\omega/2\nu}\,h(1+i)}}{e^{(1+i)\sqrt{\frac{\omega h^2}{2\nu}}} + e^{-(1+i)\sqrt{\frac{\omega h^2}{2\nu}}}}}$$

$$\boxed{C_2 = \frac{e^{+(1+i)\sqrt{\omega h^2/2\nu}}}{e^{(1+i)\sqrt{\frac{\omega h^2}{2\nu}}} + e^{-(1+i)\sqrt{\frac{\omega h^2}{2\nu}}}}}$$

$$\bar{V} = R_e\left[V e^{i\omega t}\,\frac{e^{+(1+i)\sqrt{\frac{\omega h^2}{2\nu}}\left(1-\frac{y}{h}\right)} + e^{-(1+i)\sqrt{\frac{\omega h^2}{2\nu}}\left(1-y/h\right)}}{e^{(1+i)\sqrt{\frac{\omega h^2}{2\nu}}} + e^{-(1+i)\sqrt{\frac{\omega h^2}{2\nu}}}}\right]$$

$$\frac{\omega h^2}{2\nu} \rightarrow 0: \quad \bar{V} \rightarrow R_e\left[V e^{i\omega t}\right] = V\cos(\omega t)$$

$$\frac{\omega h^2}{2\nu} \rightarrow \infty: \quad \bar{V} \rightarrow R_e\left[V e^{i\omega t} e^{-(1+i)\sqrt{\frac{\omega}{2\nu}}\,y}\right] = V e^{-\sqrt{\frac{\omega}{2\nu}}\,y}\cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}\,y\right) \rightarrow \text{STOKES}$$

LIMITING SOLUTIONS

$$\frac{\partial V_x}{\partial t} = \nu \frac{\partial^2 V_x}{\partial y^2} \quad \begin{cases} y=0: & V_x = V\cos(\omega t) \\ y=h: & \dfrac{\partial V_x}{\partial y} = 0 \end{cases}$$

$$\frac{V}{\omega^{-1}} \qquad \frac{\nu V}{h^2}$$

$$\frac{\omega h^2}{\nu} \ll 1 \rightarrow \underset{\text{SOLUTION}}{\text{Q-S}} \quad \frac{\partial^2 V_x}{\partial y^2} = 0 \quad \begin{cases} y=0: & V_x = V\cos(\omega t) \\ y=h: & \dfrac{\partial V_x}{\partial y} = 0 \end{cases} \rightarrow \boxed{V_x = V\cos(\omega t)}$$ THE LIQUID LAYER MOVES LIKE A SOLID!!

$$\frac{\omega h^2}{\nu} \gg 1 \rightarrow \frac{\partial V_x}{\partial t} = 0\ ??$$

THE MOTION IS RESTRICTED TO A THIN LAYER OF CHARACTERISTIC THICKNESS $\sqrt{\nu/\omega} \ll h \rightarrow$

$$\frac{\partial V_x}{\partial t} = \nu \frac{\partial^2 V_x}{\partial y^2} \quad \begin{cases} y=0: & V_x = V\cos(\omega t) \\ y \gg \sqrt{\nu/\omega} \rightarrow & V_x \rightarrow 0 \end{cases}$$

$$\boxed{V_x = V e^{-\sqrt{\frac{\omega}{2\nu}}\,y}\cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}\,y\right)}$$ STOKES!!