## Chapter 3

## Derivatives

### 3.1 Derivative of a function. Differentiation rules

### 3.1.1 Slope of a curve

The slope of a curve at a point $P$ is a measure of the steepness of the curve. If $Q$ is a point on the curve near $P$, then the slope of the curve at $P$ is approximately the slope of the line segment $\overline{P Q}$. The slope of the curve at $P$ is defined to be the limit of the slope of $\overline{P Q}$ as $Q$ approaches $P$ along the curve.


In symbols, slope of curve at $P=\lim _{Q \rightarrow P}($ slope of $\overline{P Q})$.
To find the slope of the curve $y=x^{2}$ at point $P=(1,1)$ we choose a point $Q$ on the curve near $P$. Let the $x$-coordinate of $Q$ be $1+h$ with $h$ small. The $y$-coordinate of $Q$ is $(1+h)^{2}$. We now calculate

$$
\text { slope of } \overline{P Q}=\frac{(1+h)^{2}-1}{(1+h)-1}=2+h
$$

As $Q$ approaches $P, h$ approaches 0 . Thus:

$$
\text { slope of curve at }(1,1)=\lim _{h \rightarrow 0}(2+h)=2
$$

Definition 3.1.1. The derivative of function $f$ at point $c, f^{\prime}(c)$, is the slope of the curve $y=f(x)$ at point $(c, f(c))$, that is:

$$
f^{\prime}(c)=\lim _{h \rightarrow 0} \frac{f(c+h)-f(c)}{h}
$$

whenever the limit exists.
We shall say that $f$ is differentiable at point $c$ if $f^{\prime}(c)$ exists.


### 3.1.2 Table of derivatives of basic elementary functions

1. $\left(x^{\alpha}\right)^{\prime}=\alpha x^{\alpha-1}$ ( $\alpha$ is any number).
2. $(\ln x)^{\prime}=\frac{1}{x}$.
3. $\left(a^{x}\right)^{\prime}=a^{x} \ln a$, in particular $\left(e^{x}\right)^{\prime}=e^{x}$.
4. $(\sin x)^{\prime}=\cos x$.
5. $(\cos x)^{\prime}=-\sin x$.
6. $(\tan x)^{\prime}=\frac{1}{\cos ^{2} x},\left(x \neq \frac{\pi}{2}+\pi n, n\right.$ integer $)$.
7. $(\arcsin x)^{\prime}=\frac{1}{\sqrt{1-x^{2}}}(-1<x<1)$.
8. $(\arccos x)^{\prime}=-\frac{1}{\sqrt{1-x^{2}}}(-1<x<1)$.
9. $(\arctan x)^{\prime}=\frac{1}{1+x^{2}}$.

### 3.1.3 The line tangent to a curve

The line tangent to a curve at a point is defined to be the line that passes through the point and that has slope that is the same as the slope of the curve at that point. Thus,

$$
y-f(c)=f^{\prime}(c)(x-c)
$$

is the the equation of the line tangent to $y=f(x)$ at point $P=(c, f(c))$.

## Example 3.1.2.

1. Find the line tangent to $y=\sqrt{x}$ at $(16,4)$.

Solution: $f(x)=x^{1 / 2}, \quad f^{\prime}(x)=\frac{1}{2} x^{-1 / 2}, \quad f^{\prime}(16)=\frac{1}{8}$.
Hence, $y-4=\frac{1}{8}(x-16)$, or $y=\frac{1}{8} x+2$.
2. Find the line tangent to $y=|x|$ at $(0,0)$.

Solution: There is no tangent line to $y=|x|$ at $(0,0)$, since the function $f(x)=|x|$ is not differentiable at 0 . To see this, notice that the limit

$$
\lim _{h \rightarrow 0} \frac{f(0+h)-f(0)}{h}=\lim _{h \rightarrow 0} \frac{|h|}{h}
$$

does not exist (the left limit is -1 and the right limit is 1 ).

### 3.1.4 One-sided derivatives

If there is the limit

$$
\lim _{h \rightarrow 0^{+}} \frac{f(c+h)-f(c)}{h} \quad\left(\lim _{h \rightarrow 0^{-}} \frac{f(c+h)-f(c)}{h}\right)
$$

then it is called the right-hand (left-hand) derivative of the function $f$ at the point $c$ and is denoted $f^{\prime}\left(c^{+}\right)\left(f^{\prime}\left(c^{-}\right)\right)$.

Theorem 3.1.3. $f^{\prime}(c)$ exists if and only if both $f^{\prime}\left(c^{+}\right)$and $f^{\prime}\left(c^{-}\right)$exists and they are equal. In this case, $f^{\prime}(c)=f^{\prime}\left(c^{+}\right)=f^{\prime}\left(c^{-}\right)$.
Example 3.1.4. Is the function $f(x)=\left\{\begin{array}{ll}x^{2}, & \text { if } x \leq 0 ; \\ x e^{-1 / x}, & \text { if } x>0 .\end{array}\right.$ differentiable at 0 ?
Solution: Yes, and $f^{\prime}(0)=0$.

$$
\begin{aligned}
& f^{\prime}\left(0^{-}\right)=\lim _{h \rightarrow 0^{-}} \frac{f(0+h)-f(0)}{h}=\lim _{h \rightarrow 0^{-}} \frac{h^{2}}{h}=0 ; \\
& f^{\prime}\left(0^{+}\right)=\lim _{h \rightarrow 0^{+}} \frac{f(0+h)-f(0)}{h}=\lim _{h \rightarrow 0^{+}} \frac{h e^{-1 / h}}{h}=e^{-\infty}=0 .
\end{aligned}
$$

### 3.1.5 Continuity and differentiability

Continuity is a necessary condition for differentiability. In other words, a discontinuous function cannot be differentiable.

Theorem 3.1.5. Let $f$ be differentiable at $c$. Then, it is continuous at $c$.

Proof. By assumption the limit

$$
f^{\prime}(c)=\lim _{h \rightarrow 0} \frac{f(c+h)-f(c)}{h}
$$

exists. We want to prove that $f$ is continuous at $c$, that is, $\lim _{h \rightarrow 0} f(c+h)=f(c)$ or, equivalently, that $\lim _{h \rightarrow 0}(f(c+h)-f(c))=0$. To this end consider the following computations:

$$
\lim _{h \rightarrow 0} \frac{h}{h}(f(c+h)-f(c))=\lim _{h \rightarrow 0} h \cdot \lim _{h \rightarrow 0} \frac{f(c+h)-f(c)}{h}=0 \cdot f^{\prime}(c)=0 .
$$

Example 3.1.6. Discuss the differentiability of the function $f(x)= \begin{cases}a x-x^{2}, & \text { if } x<1 ; \\ b(x-1), & \text { if } x \geq 1\end{cases}$ where $a, b \in \mathbb{R}$.

Solution: First we study continuity. The domain of $f$ is the whole real line. For $x<1$ and $x>1$ it is given by elementary functions, which are continuous. It remains to consider the frontier point $x=1$. We have $f(1)=0$ and $\lim _{x \rightarrow 1^{-}} f(x)=\lim _{x \rightarrow 1^{-}} a x-x^{2}=a-1$. Thus, $f$ is continuous at 1 if and only if $a=1$. Hence

If $x \neq 1$, then $f$ is continuous for any $a, b$ and at $x=1, f$ is continuous if and only if $a=1$ ( $b$ arbitrary).

Now we go with differentiability. Clearly, $f$ is differentiable at any point $x \neq 1$. When $a \neq 1 f$ is not differentiable at $x=1$ since it is not continuous at this point. Hence, let us consider $a=1$.

$$
\begin{aligned}
f^{\prime}\left(1^{-}\right)= & \lim _{h \rightarrow 0^{-}} \frac{f(1+h)-f(1)}{h}=\lim _{h \rightarrow 0^{-}} \frac{(1+h)-(1+h)^{2}-0}{h} \\
& =\lim _{h \rightarrow 0^{-}} \frac{(1+h)-\left(1+2 h+h^{2}\right)}{h}=\lim _{h \rightarrow 0^{-}} \frac{-h-h^{2}}{h}=-1 ; \\
f^{\prime}\left(1^{+}\right)= & \lim _{h \rightarrow 0^{+}} \frac{f(1+h)-f(1)}{h}=\lim _{h \rightarrow 0^{+}} \frac{b(1+h-1)}{h}=b .
\end{aligned}
$$

Hence $f^{\prime}\left(1^{-}\right)=f^{\prime}\left(1^{+}\right)=f^{\prime}(1)$ if and only if $b=-1$. In summary
If $x \neq 1$, then $f$ is differentiable for any $a, b$ and at $x=1, f$ is differentiable if and only if $a=1$ and $b=-1$.

### 3.1.6 Rules of differentiation

Let $f$ and $g$ be functions differentiable at point $c$. Then, the sum, difference, product by a scalar, product and quotient are also differentiable at $c$ and the derivatives are given by the following expressions.

1. Sum: $(f+g)^{\prime}=f^{\prime}+g^{\prime}$;
2. Difference: $(f-g)^{\prime}=f^{\prime}-g^{\prime}$;
3. Product by a scalar: $(\lambda f)^{\prime}=\lambda f^{\prime}, \lambda \in \mathbb{R}$;
4. Product: $(f \cdot g)^{\prime}=f^{\prime} \cdot g+f \cdot g^{\prime}$;
5. Quotient: $\left(\frac{f}{g}\right)^{\prime}=\frac{f^{\prime} g-f g^{\prime}}{g^{2}}, g(c) \neq 0$.

### 3.1.7 Chain Rule

(Derivative of a compose function). Let $f$ be differentiable at $c$ and let $g$ be differentiable at $f(c)$. Then the composition $g \circ f$ is differentiable at $c$ and the derivative

$$
(g \circ f)^{\prime}(c)=g^{\prime}(f(c)) \cdot f^{\prime}(c) .
$$

Example 3.1.7. Find the derivative of $y=\sin \left(3^{x}+x^{3}\right)$.
Solution: We can represent the function in the form $y=\sin t$ where $t=3^{x}+x^{3}$. Using the chain rule we get

$$
y^{\prime}=\left.(\sin t)^{\prime}\right|_{t=3^{x}+x^{3}}\left(3^{x}+x^{3}\right)^{\prime}=\cos \left(3^{x}+x^{3}\right)\left(3^{x} \ln 3+3 x^{2}\right) .
$$

Example 3.1.8. Find the derivative of $h(x)=\sqrt{e^{x}-x^{2}}$ at the point $c=1$.
Solution:

$$
h^{\prime}(x)=\left(\left(e^{x}-x^{2}\right)^{\frac{1}{2}}\right)^{\prime}=\frac{1}{2}\left(e^{x}-x^{2}\right)^{-\frac{1}{2}}\left(e^{x}-x^{2}\right)^{\prime}=\frac{1}{2}\left(e^{x}-x^{2}\right)^{-\frac{1}{2}}\left(e^{x}-2 x\right) .
$$

Hence $h^{\prime}(1)=\frac{e-2}{2 \sqrt{e-1}}$, approximately 0.274
Example 3.1.9. In the following examples it is supposed that $f$ is a differentiable function.

- $\left(e^{f(x)}\right)^{\prime}=f^{\prime}(x) e^{f(x)}$;
- $\left(a^{f(x)}\right)^{\prime}=(\ln a) f^{\prime}(x) a^{f(x)}$.
- $(\ln f(x))^{\prime}=\frac{f^{\prime}(x)}{f(x)}$;
- $(\arctan f(x))^{\prime}=\frac{f^{\prime}(x)}{1+f^{2}(x)}$.


### 3.1.8 Derivative of the inverse function

Let $f$ be continuous and one-to-one in an open interval $(x-\delta, x+\delta)$ and such that $f^{\prime}(x)$ exists. Then $f^{-1}$ is differentiable at $y=f(x)$ and the derivative is

$$
\begin{equation*}
\left(f^{-1}(y)\right)^{\prime}=\frac{1}{f^{\prime}(x)}=\frac{1}{f^{\prime}\left(f^{-1}(y)\right)} . \tag{3.1.1}
\end{equation*}
$$

The proof of this assertion is easy using the Chain Rule. For, deriving in both sides of the identity

$$
x=f^{-1}(f(x)),
$$

we obtain

$$
1=\left(f^{-1}\right)^{\prime}(f(x)) \cdot f^{\prime}(x)
$$

The formula follows, once we substitute $y=f(x)$.
Example 3.1.10. Prove that $\arctan ^{\prime} x=\frac{1}{1+x^{2}}$.
Solution: The function $\arctan x$ is the inverse of the function $\tan x$. According to the formula (3.1.1) above

$$
\arctan ^{\prime} y=\frac{1}{1+\tan ^{2} x},
$$

because $\tan ^{\prime} x=1+\tan ^{2} x$, and where $y=\tan x$. Thus,

$$
\arctan ^{\prime} y=\frac{1}{1+y^{2}}
$$

Of course, we can change the name of the variable from $y$ to $x$ to get the result.

### 3.1.9 Using the derivative for approximations

The line tangent to a curve at a point $(c, f(c))$ coincides with the curve at the point of tangency, and constitute a good approximation of the curve at points near $(c, f(c))$. In fact, a function is differentiable at a point when the graph of the function at this point can be well approximated by a straight line (the tangent line).

Thus, for small values of $h$, the value of $f(c+h)$ can be approximated by the known valued of $f(c)$ and $f^{\prime}(c)$ :

$$
\begin{equation*}
f(c+h) \approx f(c)+f^{\prime}(c) h \tag{3.1.2}
\end{equation*}
$$

Example 3.1.11. Without using a pocket calculator, give an approximated value of $\sqrt{0.98}$.

Solution: Let us consider the function $f(x)=\sqrt{1+x}$. Notice that $f(0)=1$, $f(-0.02)=\sqrt{0.98}, f^{\prime}(x)=\frac{1}{2}(1+x)^{-1 / 2}, f^{\prime}(0)=0.5$. We find from formula (3.1.2) with $c=0$ and $h=-0.02$ that

$$
\sqrt{0.98}=f(0-0.02) \approx f(0)+f^{\prime}(0)(-0.02)=1+0.5(-0.02)=0.99
$$

### 3.1.10 Implicit differentiation

Definition 3.1.12. An equation $F(x, y)=0$ defines $y=f(x)$ in an implicit way near the point $\left(x_{0}, y_{0}\right)$ when it is satisfied that:

1. $F\left(x_{0}, y_{0}\right)=0$
2. if $(x, y)$ is close to the point $\left(x_{0}, y_{0}\right): F(x, y)=0 \Longleftrightarrow y=f(x)$.

Theorem 3.1.13. $F\left(x_{0}, y_{0}\right)=0, \frac{\partial F}{\partial y}\left(x_{0}, y_{0}\right) \neq 0 \Longrightarrow$
The equation $F(x, y)=0$ defines $y=f(x)$ in an implicit way near the point $\left(x_{0}, y_{0}\right)$.
Theorem 3.1.14. $y_{0}^{\prime}=f^{\prime}\left(x_{0}\right)$ can be obtained from the equation:
$\frac{\partial F}{\partial x}\left(x_{0}, y_{0}\right)+\frac{\partial F}{\partial y}\left(x_{0}, y_{0}\right) y_{0}^{\prime}=0$
This way, even if we do not know the explicit expression of $y=f(x)$, we can have an approximate idea of the function knowing that
$y-y_{0}=f^{\prime}\left(x_{0}\right)\left(x-x_{0}\right)$
is the tangent line of $f(x)$ at the point $\left(x_{0}, y_{0}\right)$.
Besides, if by taking the derivative of the equation $(*)$ we find $y_{0} " \neq 0$, our information on the function improves since:

1. if $y_{0} ">0 \Longrightarrow f$ is convex near $x_{0} \Longrightarrow$ the graph of $f$ lies above the tangent line near the point $\left(x_{0}, y_{0}\right)$.
2. if $y_{0} "<0 \Longrightarrow f$ is convex near $x_{0} \Longrightarrow$ the graph of $f$ lies below the tangent line near the point $\left(x_{0}, y_{0}\right)$.

### 3.2 Some Theorems on Differentiable Functions

### 3.2.1 Monotonicity

The function $f$ is said to be increasing at a point $c$ if there is some interval around the point $c$ in which $f(x)>f(c)$ for $x>c, f(x)<f(c)$ for $x<c$.

A decrease of a function at a point can be defined analogously.
For example, $c=0$ is a point of increasing of $x^{3}$, but is not of $x^{2}$.
Theorem 3.2.1. If the function $f$ is differentiable at $c$ and $f^{\prime}(c)>0\left(f^{\prime}(c)<0\right)$, then $f$ increases (decreases) at the point $c$.

The theorem establishes only a sufficient condition, because $c=0$ is a point of increasing of $f(x)=x^{3}$ but $f^{\prime}(0)=0$.

The following results are about the monotonic behavior of a differentiable function in an interval.

Theorem 3.2.2. For the function $f$ differentiable on an interval I to be increasing (decreasing) it is necessary and sufficient that for every $x \in I f^{\prime}(x) \geq 0\left(f^{\prime}(x) \leq 0\right)$ holds.

Theorem 3.2.3. If $f^{\prime}(x)>0\left(f^{\prime}(x)<0\right)$ for every $x \in I$, then $f$ is strictly increasing (decreasing) on the interval I.

Example 3.2.4. Find the intervals of monotonicity of $f(x)=3 x-x^{3}$.
Solution: We have $f^{\prime}(x)=3-3 x^{2}=3\left(1-x^{2}\right)$. Since $f^{\prime}(x)>0$ for $x \in(-1,1)$ and $f^{\prime}(x)<0$ for $x \in(-\infty, 1)$ and $x \in(1,+\infty)$, it follows that $f$ is strictly increasing in $[-1,1]$ and strictly decreasing in $(-\infty,-1] \cup[1, \infty)$.

### 3.2.2 Local Extremum of functions

Derivatives are a very useful tool for locating and identifying maximum and minimum values (extremum) of functions. In the following, we suppose that the function $f$ is defined in an open interval $(c-\delta, c+\delta)$ around $c$.
Definition 3.2.5. The function $f$ has a local maximum (minimum) at the point $c$ if there is $\delta>0$ such that for every $x \in(c-\delta, c+\delta)$

$$
f(x) \leq f(c) \quad(f(x) \geq f(c))
$$

A local maximum or a local minimum are local extremum of $f$.
Theorem 3.2.6. If the function $f$ has an extremum at the point $c$, then the derivative $f^{\prime}(c)$ is either zero or does not exist.
Proof. Without loss of generality, suppose that $c$ a local minimum point of $f$ and that $f^{\prime}(c)$ exists. By definition of a local minimum, we have $f(c+h) \geq f(c)$ for every $h$ with $|h|<\delta$. Let $h>0$ and consider the quotient

$$
\frac{f(c+h)-f(c)}{h} .
$$

It is non-negative and the limit exists when $h \rightarrow 0$ and equals $f^{\prime}(c)$, since $f$ is differentiable at $c$. Given that the limit of non-negative quantities must be non-negative, we get the inequality $f^{\prime}(c) \geq 0$. Consider now $h<0$. Then, the above quotient is non-positive. Taking the limit as $h \rightarrow 0$ we get the reverse inequality $f^{\prime}(c) \leq 0$. Hence it must be $f^{\prime}(c)=0$ and we are done.

The points where the function is not differentiable or the derivative vanishes are possible extrema of $f$, and for this reason they are called critical points of $f$.
Example 3.2.7. Find the critical points of $f(x)=3 x-x^{3}$ and $g(x)=|x|$.
Solution: Function $f$ is differentiable at every point and $f^{\prime}(x)=3\left(1-x^{2}\right)$. Thus, $f^{\prime}(x)=0$ if and only if $x= \pm 1$. Hence, the critical points of $f$ are 1 and -1 . Function $g$ is differentiable at every point except $c=0$, where it has a corner. Actually, the derivative

$$
g^{\prime}(x)= \begin{cases}1, & \text { if } x>0 \\ -1, & \text { if } x<0\end{cases}
$$

never vanishes. Hence, 0 is the only critical point of $g$.
Theorem 3.2.8. Suppose that $f$ is differentiable in an interval $I=(c-\delta, c+\delta)$ around point c (except, maybe, for the point c itself). Then, if the derivative of $f$ changes sign from plus to minus (from minus to plus) when passing through the point $c$, then $f$ has a local maximum (minimum) at the point c. If the derivative does not change sign when passing through the point $c$, then the function $f$ does not posses an extremum at the point $c$.
Example 3.2.9. Find the the local extrema points of $f(x)=3 x-x^{3}$ and $g(x)=|x|$.
Solution: We know from Example 3.2.4 that the sign of $f^{\prime}$ change from minus to plus at -1 and from plus to minus at 1 , hence $f$ has a local minimum at -1 , and $f$ has a local maximum at 1 . On the other hand, $g^{\prime}$ change from minus to plus at 0 (see Example 3.2.7), hence although $g$ is not differentiable at $0, g$ has a local minimum at this point.

### 3.2.3 Theorems of Rolle and Lagrange

Theorem 3.2.10 (Rolle's Theorem). Let the function $f$ satisfy the following conditions:

1. $f$ is continuous on $[a, b]$;
2. $f$ is differentiable in $(a, b)$;
3. $f(a)=f(b)$.

Then there is a point $c \in(a, b)$ such that $f^{\prime}(c)=0$.
Rolle's Theorem states that there is a point $c \in(a, b)$ such that the tangent line to the graph of the function $f$ at the point $(c, f(c))$ is parallel to the $x$-axis.

Theorem 3.2.11 (Lagrange's Theorem). Let the function $f$ satisfy the following conditions:

1. $f$ is continuous on $[a, b]$;
2. $f$ is differentiable in $(a, b)$.

Then there is a point $c \in(a, b)$ such that

$$
f(b)-f(a)=f^{\prime}(c)(b-a) .
$$

Lagrange's Theorem is also known as Mean Value Theorem. It can be interpreted as follows: The number

$$
\frac{f(b)-f(a)}{b-a}
$$

is the slope of the line $r$ which passes through the points $(a, f(a))$ and $(b, f(b))$ of the graph of $f$, and $f^{\prime}(c)$ is the slope of the tangent to the graph of $f$ at $(c, f(c))$. Lagrange's formula shows that this tangent line is parallel to the straight line $r$.

### 3.3 L'Hopital's Rule

Now we present a useful technique for evaluating limits that uses the derivatives of the functions involved.

Theorem 3.3.1 (Indetermination of the type 0/0). Assume that the following conditions are fulfilled:

1. The functions $f$ and $g$ are defined and differentiable in an interval $I=(c-\delta, c+\delta)$ around point $c$ (except, maybe, the point $c$ itself);
2. $\lim _{x \rightarrow c} f(x)=\lim _{x \rightarrow c} g(x)=0$;
3. The derivative $g^{\prime}(x) \neq 0$ for any $x \in I$ (except, maybe, the point $c$ itself).
4. There exists the limit $\lim _{x \rightarrow c} \frac{f^{\prime}(x)}{g^{\prime}(x)}$.

Then,

$$
\lim _{x \rightarrow c} \frac{f(x)}{g(x)}=\lim _{x \rightarrow c} \frac{f^{\prime}(x)}{g^{\prime}(x)} .
$$

Example 3.3.2. Evaluate $\lim _{x \rightarrow 0} \frac{\sin a x}{\tan b x}$, where $a, b \in \mathbb{R}$.
Solution: The limit is of the indeterminate type $0 / 0$. It is easy to verify that all conditions of Theorem 3.3.1 are fulfilled. Consequently,

$$
\lim _{x \rightarrow 0} \frac{\sin a x}{\tan b x}=\lim _{x \rightarrow 0} \frac{a \cos a x}{\frac{b}{\cos ^{2} b x}}=\frac{a}{b} .
$$

Theorem 3.3.3 (Indetermination of the type $\pm \infty / \infty$ ). Assume that (1), (3) and (4) of Theorem 3.3.1 are fulfilled and that (2) is replaced by
(2') $\lim _{x \rightarrow c} f(x)=\lim _{x \rightarrow c} g(x)= \pm \infty$.
Then,

$$
\lim _{x \rightarrow c} \frac{f(x)}{g(x)}=\lim _{x \rightarrow c} \frac{f^{\prime}(x)}{g^{\prime}(x)} .
$$

Example 3.3.4. Evaluate $\lim _{x \rightarrow 0^{+}} x \ln x$.
Solution: The limit is of the indeterminate type $0 \cdot \infty$. Writing $x \ln x$ as $\frac{\ln x}{1 / x}$ we get an indeterminate form $\infty / \infty$. Applying L'Hospital Rule one obtains

$$
\lim _{x \rightarrow 0^{+}} \frac{\ln x}{1 / x}=\lim _{x \rightarrow 0^{+}} \frac{1 / x}{-1 / x^{2}}=\lim _{x \rightarrow 0^{+}}(-x)=0 .
$$

Remark 3.3.5. Similarly to Theorems 3.3 .1 and 3.3.3, L'Hopital's Rule can be also stated when $x \rightarrow+\infty$ or $x \rightarrow-\infty$,

$$
\lim _{x \rightarrow \pm \infty} \frac{f(x)}{g(x)}=\lim _{x \rightarrow \pm \infty} \frac{f^{\prime}(x)}{g^{\prime}(x)} .
$$

Example 3.3.6. Evaluate $\lim _{x \rightarrow \infty} \frac{\ln x}{x}$.
Solution: The limit is of the indeterminate type $\infty / \infty$. Differentiating above and below, we get

$$
\lim _{x \rightarrow \infty} \frac{\ln x}{x}=\lim _{x \rightarrow \infty} \frac{1 / x}{1}=0 .
$$

Remark 3.3.7. Indeterminate forms of other types, $0 \cdot \infty, \infty-\infty, 1^{\infty}, 0^{0}$ or $\infty^{0}$ can be reduced to indeterminate forms of type $0 / 0$ or $\infty / \infty$ and to them we can apply L'Hopital's Rule.
Example 3.3.8. Evaluate $\lim _{x \rightarrow \infty} x^{1 / x}$.
Solution: The limit is of the indeterminate type $\infty^{0}$. We represent $x^{1 / x}=e^{\ln x / x}$ and study $\lim _{x \rightarrow \infty} \frac{\ln x}{x}=\lim _{x \rightarrow \infty} \frac{1}{x}=0$, hence the limit is $e^{0}=1$.

Example 3.3.9. Evaluate $\lim _{x \rightarrow 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$.
Solution: The limit is of indeterminate type $\infty-\infty$. If we combine fractions, then we obtain

$$
\frac{x \ln x-(x-1)}{(x-1) \ln x}
$$

which is of the form $0 / 0$ at $x=1$. Differentiating above and below, we get

$$
\frac{\ln x}{1-x^{-1}+\ln x}
$$

which is again $0 / 0$ at $x=1$. Another differentiation above and below gives

$$
\frac{x^{-1}}{x^{-2}+x^{-1}}=\frac{x}{1+x}
$$

which has limit $1 / 2$ as $x \rightarrow 1$. Thus, we had to apply L'Hospital Rule twice to find that the limit is $1 / 2$.

When the hypotheses of the theorems are not satisfied, we might obtain incorrect answers, as in the following example.
Example 3.3.10. Clearly, $\lim _{x \rightarrow 0^{+}} \frac{\ln x}{x}=\frac{-\infty}{0^{+}}=-\infty$. If we attempt to use L'Hopital Rule, we would obtain

$$
\lim _{x \rightarrow 0^{+}} \frac{\ln x}{x}=\lim _{x \rightarrow 0^{+}} \frac{x^{-1}}{1}=+\infty
$$

which is incorrect. Notice that the limit is not indeterminate, hence theorems 3.3.1 and 3.3.3 do not apply.

### 3.4 Optimization of continuous functions on intervals $[a, b]$

Consider a continuous function $f$ defined on an interval $I=[a, b]$. By Weierstrass' Theorem, $f$ attains in $[a, b]$ global extremum. On the other hand, since a global extremum is also a local extremum, if the global extremum are in the open interval $(a, b)$, then they must be critical points of $f$. Hence, to locate and classify the global extremum of $f$, we will use the following recipe:

1. Find the critical points of $f$ in $(a, b)$;
2. Evaluate $f$ at the critical points found in (a) and at the extreme points of the interval, $a, b$;
3. Select the maximum value (global maximum) and the minimum value (global minimum).

Example 3.4.1. Find and classify the extremum points of $f(x)=3 x-x^{3}$ in the interval $[-2,2]$.

Solution: Since $f$ is continuous and $I=[-2,2]$ is closed and bounded, by Weierstrass' Theorem $f$ attains on $I$ global maximum and minimum. Then, as explained above, the possible global extremum are among the critical points of $f$ in $I$ and the extreme points of the interval $I:-2$ and 2 . We know that $-1 \in I$ is a local minimizer, $f(-1)=-2$, and $1 \in I$ is a local maximizer, $f(1)=2$, see Example 3.2.9. On the other hand, $f(-2)=2$ and $f(2)=-2$, thus points -1 and 2 are both global minimizers of $f$ in $I$, and points -2 and 1 are global maximizers of $f$ in $I$.

