Fundamental Concepts of Statistics

Exercise session 2

1. The speed of a molecule in a uniform gas at equilibrium is a random variable V whose density function is given by

$$f_V(v) = av^2 e^{-\frac{m}{2kT}v^2},\ v > 0,$$

with a a constant, k denoting Boltzmann's constant, T the absolute temperature and m the mass of a molecule.
a) Derive the distribution of the kinetic energy $W = \frac{mV^2}{2}$
b) Compute $E(W)$

2. Find the density of $Y = X^2$ where X is uniformly distributed on the interval $[-1, 1]$.

3. Let $f(x) = \alpha x^{-\alpha-1}$ for $x > 1$ and $f(x) = 0$ otherwise, with $\alpha > 0$. Show how to generate random variables from this density from a uniform random number generator.

4. The Weibull cdf is given by

$$F(x) = 1 - e^{-(x/\alpha)^\beta},\ x > 0,\ \alpha, \beta > 0.$$

a) Find the density function.
b) Show that if W follows a Weibull distribution then $X = (W/\alpha)^\beta$ follows an exponential distribution.
c) How to generate random variables from this density from a uniform random number generator ?
5. Compute $Cov(X + Y, X - Y)$, where X and Y are random variables with equal variances.

6. If X and Y are independent random variables and $Z = Y - X$ find expressions for the covariance and the correlation of X and Z in terms of the variances of X and Y.

7. Two independent measurements X and Y are taken from a quantity μ. We assume $E(X) = E(Y) = \mu$, but σ_X and σ_Y are unequal. The two

measurements are combined in a weighted average

$$Z = \alpha X + (1 - \alpha)Y,$$

where $0 \leq \alpha \leq 1$.

Show that $E(Z) = \mu$.

Find α in terms of σ_X and σ_Y to minimize $Var(Z)$.