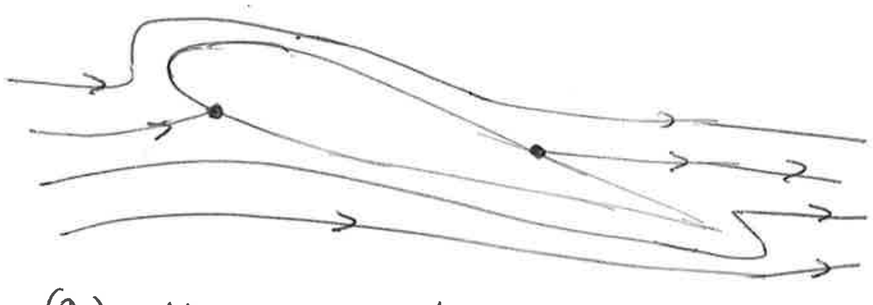
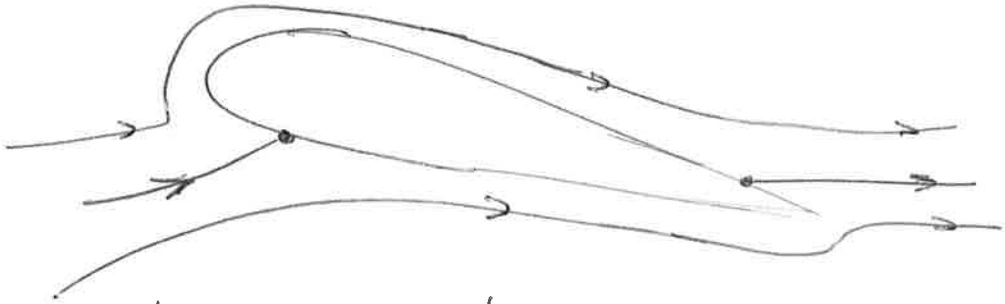
Incompressible flow over Airfoils

Kutta- Condition

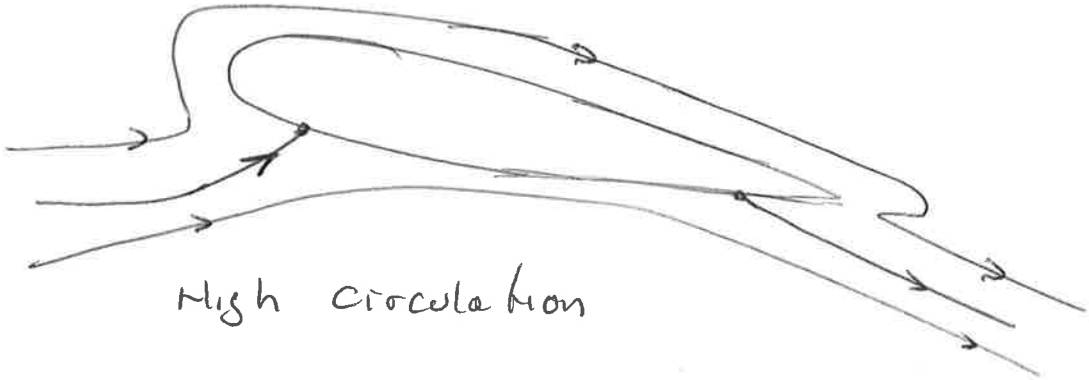
- For a given airfoil at a given angle of attack there are infinite number of valid theorical solutions, corresponding to an infinite choice of Γ. We know ~~from~~ from experience that a given airfoil at a given angle of attack produces a single value of Lift. So, although there is an infinite number of possible potential flow solutions, nature knows how to pick a particular solution. We need an additional condition that fixes Γ for a given airfoil at a given α.



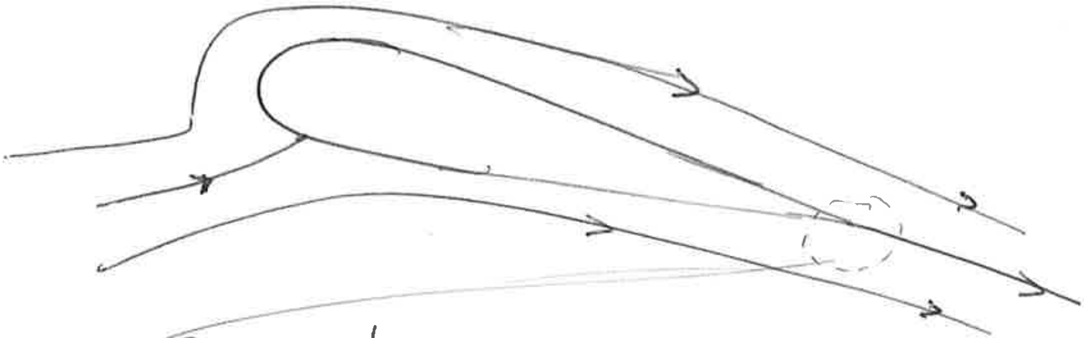
(a) No circulation



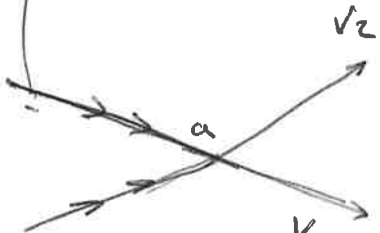
Low circulation



High circulation

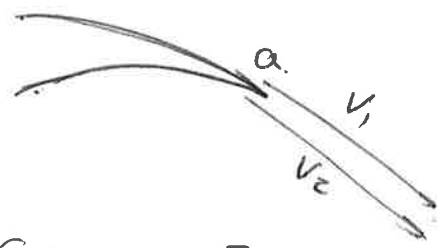


Circulation such that ~~K~~Kutta condition satisfied.



For finite trailing ~~e~~edge angle, if these velocities were finite at point a. then, there will be two different velocities of different directions at same point.

This is not possible, and only recourse is for both V_1 and V_2 to be zero at a. That is, for the finite trailining edge, point a is a stagnation point, $V_1 = V_2 = 0$.



Cusped TE.

V_1 and V_2 are in the same direction at point a and hence both V_1 and V_2 can be finite. But the pressure at point a, is a single unique value.

$$P_a + \frac{1}{2}\rho V_1^2 = P_a + \frac{1}{2}\rho V_2^2 \quad \text{Bernoulli equation.}$$

$V_1 = V_2$ (Both top and bottom surfaces velocities of the airfoil at TE are finite and equal in magnitude and direction.

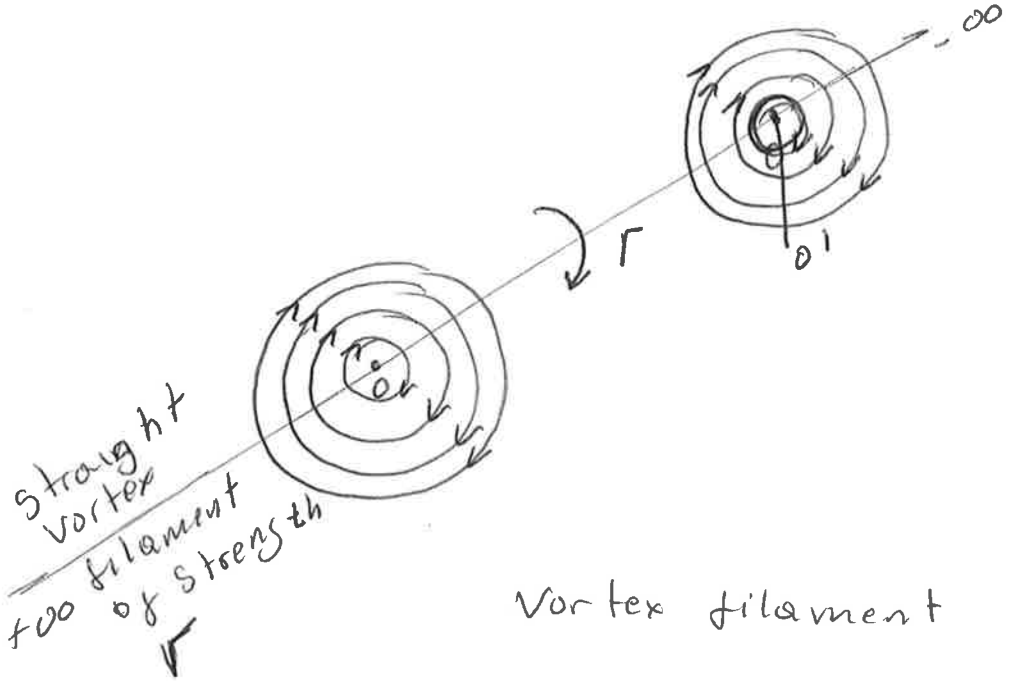
The statement of the Kutta condition as follows

1. For a given airfoil at a given angle of attack. Value of Γ around the airfoil is such that the flow leaves the trailing edge smoothly
2. If the trailing-edge angle is finite, then the trailing edge is a stagnation point

3. If the trailing edge is cusped, then the velocities leaving the top and bottom surfaces at the trailing edge are finite and equal in magnitude and direction (removes the second stagnation point).

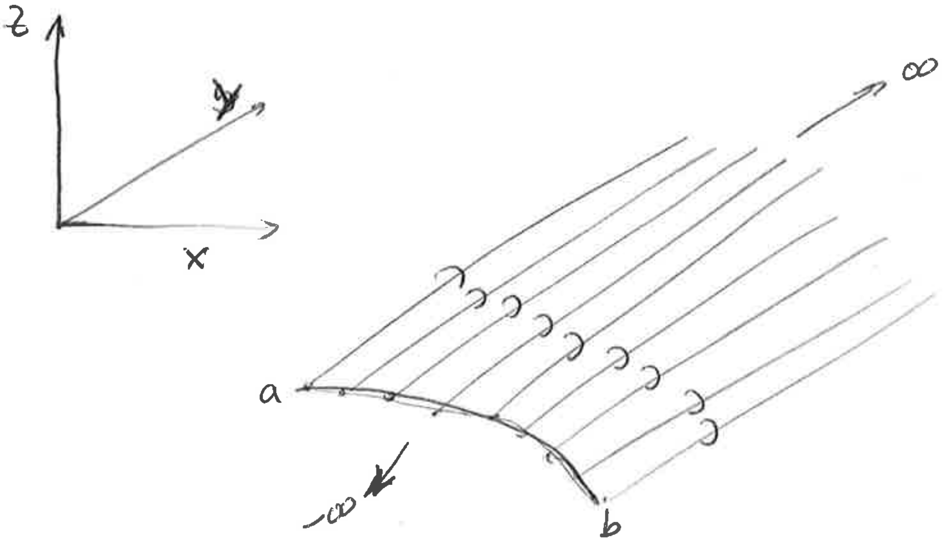
* Theoretical solutions for low-speed flow over airfoils the vortex sheet.

Consider a flow where all the streamlines are concentric circles about a given point, moreover, let the velocity along any given circular streamline be constant, but let it vary from one streamline to another inversely with distance from the common center, this flow is called a vortex flow.

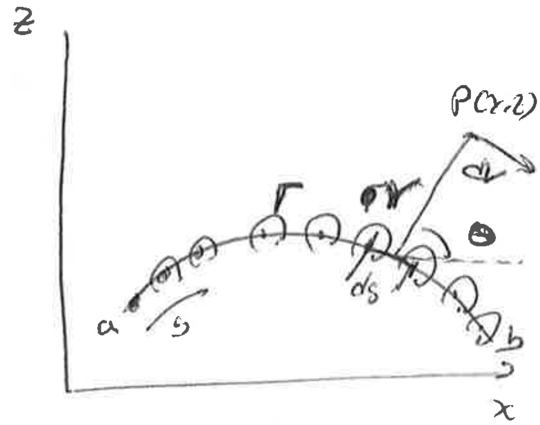


Vortex filament

We already discussed the concept of source sheet, which is an infinite number of line sources side by side. with the strength of each line source being infinitesimally small.



Vortex sheet in perspective

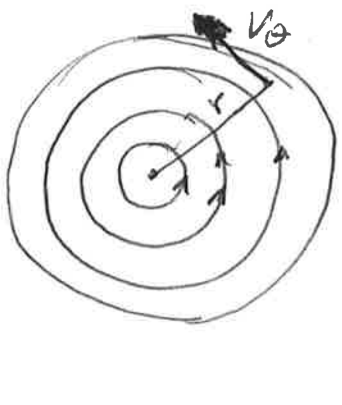


Edge view of sheet.

γ is defined as the strength of the vortex sheet per unit length along s $\gamma = \gamma(s)$, for a ds the strength is γds. The small section of the vortex sheet of strength γds induces an infinitesimally small dV at point P.

$$dV = -\frac{\gamma ds}{2\pi r}$$

This is obtained by follows



Vortex flow

From definition of vortex flow

$$V_\theta = \frac{c}{r}$$

To evaluate the constant C. Have the circulation around a given circular streamline of radius r.

$$\Gamma = -\oint_C \vec{V} \cdot dS = -V_\theta \cdot (2\pi r)$$

$$V_\theta = -\frac{\Gamma}{2\pi r}$$

$$V_\theta = \frac{C}{r} = \frac{-\Gamma}{2\pi r} \Rightarrow C = -\frac{\Gamma}{2\pi} \Rightarrow \Gamma = -2\pi C$$

It is the strength of the vortex.

It is more convinient to deal with the velocity potential.

$$\left[d\phi = -\frac{\gamma ds}{2\pi} \theta \right]$$

This is obtained by follows. velocity potential vortex

$$\frac{d\phi}{dr} = V_r = 0$$

$$\frac{1}{r}\frac{d\phi}{d\theta} = V_\theta = \frac{-\Gamma}{2\pi r}$$

$$d\phi = \frac{-\Gamma}{2\pi \cancel{r}} \cancel{r}\, d\theta =$$

$$\phi = \frac{-\Gamma}{2\pi} \theta$$

$$d\phi = -\frac{\gamma\, ds}{2\pi}\,\theta$$

The velocity potential at p due to entire vortex sheet from a to b is

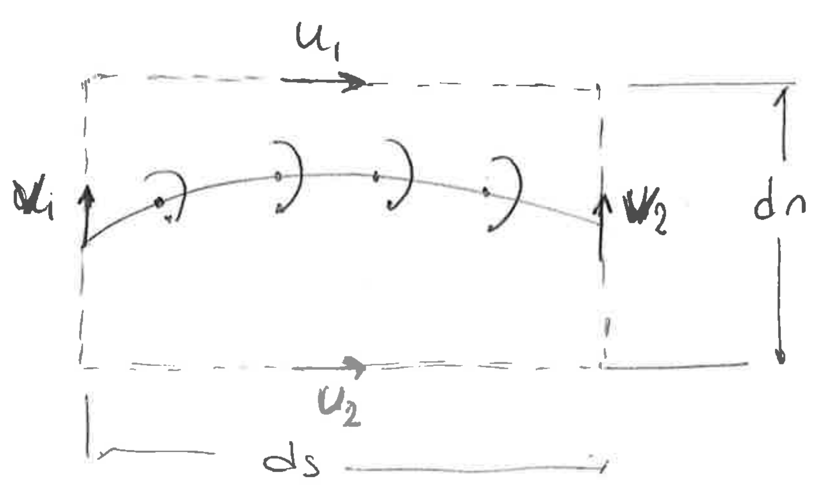
$$\phi(x,z) = -\frac{1}{2\pi}\int_a^b \theta\,\gamma\, ds$$

This is important for the numerical vortex panel method.

The circulation around the vortex sheet in above figure is the sum of the ~~s~~ strength of elemental vortices

$$\Gamma = \int_a^b \gamma\, ds$$

In vortex sheet, there is a discontinuos change in the tangential component of velocity across the sheet, whereas the normal component of velocity is preserved across the sheet. This change of tangential velocity across the vortex sheet is related to the strength of the sheet as follows



by the definition of circulation

$$\Gamma = -(V_2\,dn - u_1\,ds - V_1\,dn + u_2\,ds)$$

$$\Gamma = (u_1 - u_2)\,ds + (V_1 - V_2)\,dn \quad \text{and}$$

$$\Gamma = \gamma\,ds = (u_1 - u_2)\,ds + (V_1 - V_2)\,dn$$

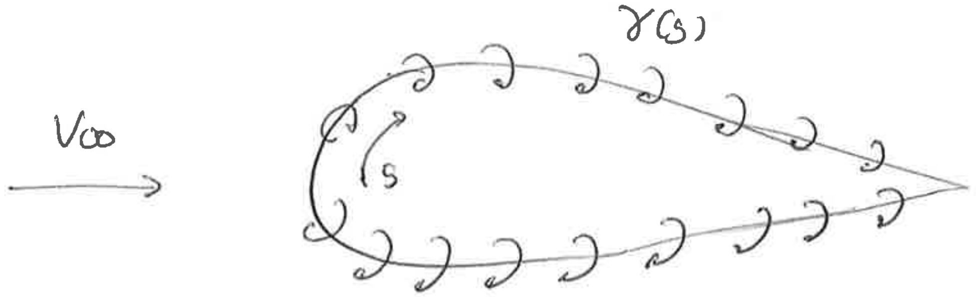
Let $dn \to 0$. In the limit u_1 and u_2 become the velocity tangential to the vortex sheet immediately above and below the sheet.

$$\gamma\,ds = (u_1 - u_2)\,ds$$

$$\gamma = (u_1 - u_2)$$

The local jump in tangential velocity across the vortex sheet is equal to the local sheet strength.

Consider a airfoil of arbitrary shape and thikness in a freestream with velocity V_∞. Replacing the airfoil surface with a Vortex sheet of variable $\gamma(s)$. Calculate the variation of γ as a function of s such that the induced velocity field from the vortex sheet when added to uniform velocity magnitude V_∞ will make vortex sheet a streamline of the flow.



The circulation around the airfoil will be given by

$$\Gamma = \int \gamma \, ds$$

The integral is taken around the complete surface of the airfoil. Finally the lift is given by Kutta-Joukowski theorem.

$$L = \rho_\infty \cdot V_\infty \Gamma$$

The concept of replacing the airfoil surface with a vortex sheet is more than just mathematical device. In real case, there is a thin boundary layer on the surface due to the action of friction. This boundary layer is a highly viscous region in which the large velocity gradients produce substantial vorticity, that $\nabla \times \vec{V}$ is finite within the boundary layer.

There is a distribution of vorticity along the airfoil surface due to viscous effects, and replacing the airfoil surface with vortex sheet can be constructed as a way of modeling this effect in a inviscid flow.

Consider the method of simulating the airfoil with vortex sheets placed either on the surface or on the chamber line.. The strength of such a vortex sheet is variable along the sheet and is defined by $\gamma(s)$. The statement of Kutta condition in term of the vortex sheet is

$$\gamma(TE) = \gamma(a) = V_1 - V_2$$

However, for the finite angle trailing edge $V_1 = V_2 = 0 \Rightarrow \gamma(TE) = 0$.

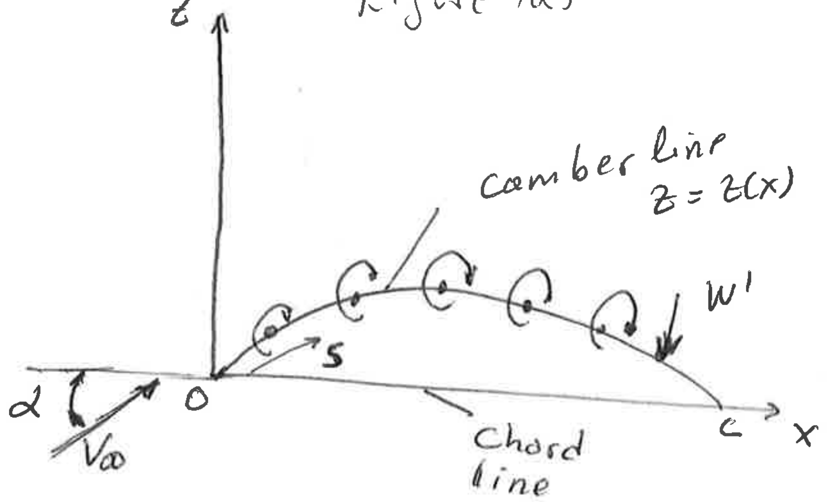
For cusped trailing edge $V_1 = V_2 \neq 0$. this leads to $\gamma(TE) = 0$.

Classical thin Airfoil theory the Symmetric Airfoil.

The basic equations necessary for the calculation of airfoil lift and moment are discussed base on the thin airfoil (Symmetric)

Consider a Vortex sheet placed on the camber line of an airfoil. The stream flow velocity is V_∞ and α (angle of attack).

Figure (a)



w': Is the normal Component to the camber line induced by the vortex sheet. $w'(s)$

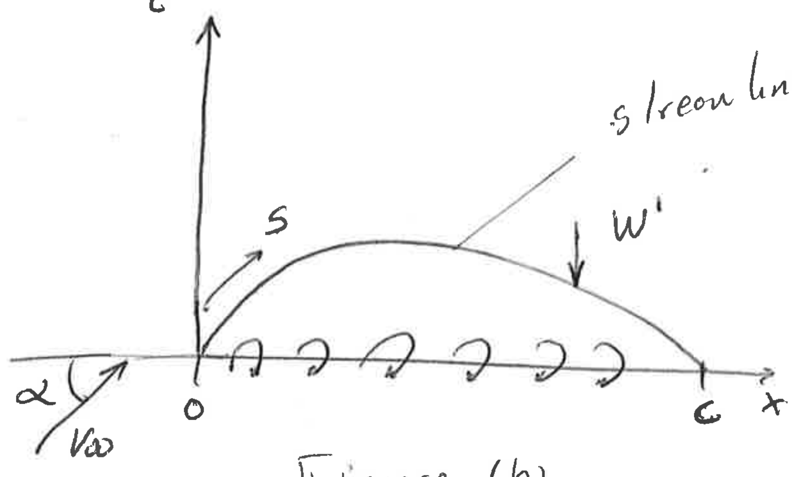


Figure (b)

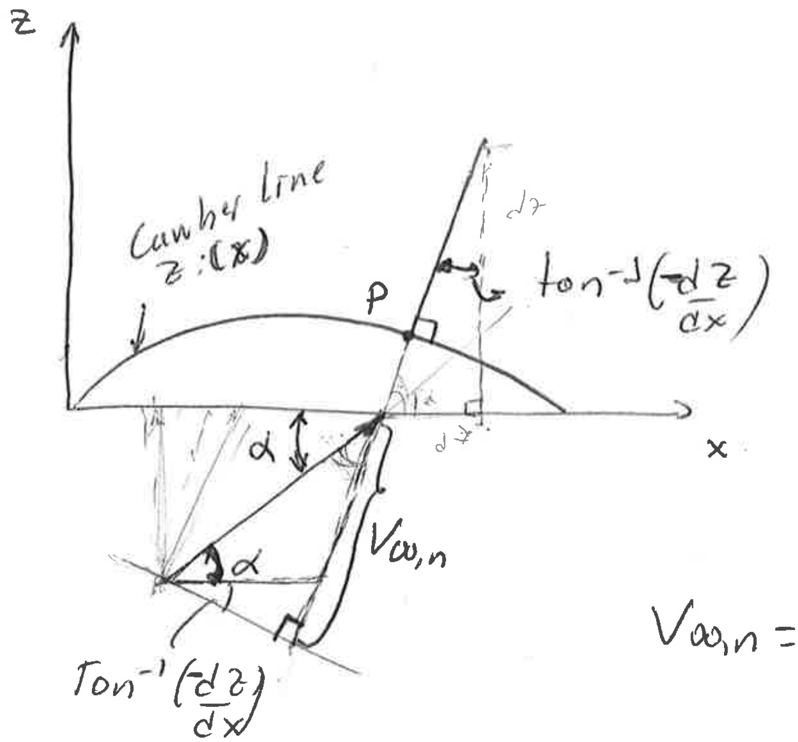
$\gamma : \gamma(x)$ and also satisfied the Kutta Condition $\gamma(c) = 0$

For the camber line to be streamline, the component of velocity normal to the camber line must be zero at all points along the camber line.

The velocity at any point is the sum of the uniform freestream velocity and the induced by the vortex sheet.

$$V_{\infty,n} + w'(s) = 0$$ at every point along the camber line.

$$V_{\infty,n} = V_\infty \cdot \sin\left[\alpha + \tan^{-1}\left(-\frac{dz}{dx}\right)\right]$$



For a thin airfoil at small angle of attack both α and $\left(-\frac{dz}{dx}\right)$ are small values.

$$\sin\theta \cong \tan\theta \cong \theta$$

$$V_{\infty,n} = V_\infty\left(\alpha - \frac{dz}{dx}\right)$$

Let's develop an expression for $w'(s)$ in terms of the strength of the vortex sheet.

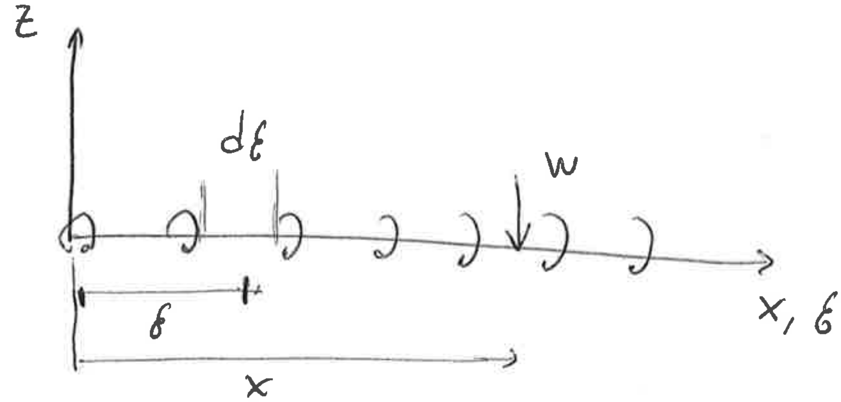
Let $w(x)$ denote the component of the normal velocity to the chord line induced by vortex sheet, as also shown in the above figure (b); because the airfoil is thin, we can approximate

$$w'(s) = w(x)$$

$w(x)$ can be obtained by using the equation of

$$dv = -\frac{\gamma\, ds}{2\pi r}$$

We need to approximate the camber too thin.



we need to determine the value of w at x location.

$$dw = -\frac{\gamma(\xi)\, d\xi}{2\pi(x-\xi)}$$

$\begin{cases} \xi = 0 & LE \\ \xi = c & TE \end{cases}$

$$w(x) = -\int_0^c \frac{\gamma(\xi)\, d\xi}{2\pi(x-\xi)} = w'(s)$$

We know $V_{\infty,n} + w'(s) = 0$

$$V_\infty\left(\alpha - \frac{dz}{dx}\right) - \int_0^c \frac{\gamma(\xi)\, d\xi}{2\pi(x-\xi)}$$

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)\,d\xi}{x-\xi} = V_\infty\left(\alpha - \frac{dz}{dx}\right)$$

The fundamental equation of thin airfoil theory. It is simply a statement that the camber line is a streamline of the flow. The only unknown is the strength $\gamma(\xi)$.

For $\gamma(\xi)$ at chord yield $\gamma(c)=0$ to satisfy the Kutta-condition. For the case of symmetric airfoil the camber line is coincident with the chord line. $dz/dx = 0$

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)\,d\xi}{(x-\xi)} = V_\infty \alpha.$$

this is an expression for a flat plate: It also is an exact expression for inviscid, incompressible flow over a flat plate. at small angle of attack.

We use the transformation ξ into θ via the following transformation

$$\xi = \frac{c}{2}(1-\cos\theta).$$

x is a fixed point in the equation, it corresponds to a particular value of θ. such θ_0

$$x = \frac{c}{2}(1-\cos\theta_0)$$

$$d\xi = \frac{c}{2} \sin\theta \, d\theta$$

$$\frac{1}{2\pi} \int_0^{\pi} \frac{\gamma(\theta) \cdot \sin\theta \, d\theta}{\cos\theta - \cos\theta_0} = V_\infty \cdot \alpha.$$

The solution of the integror is

$$\left[\gamma(\theta) = 2\alpha V_\infty \left(\frac{1 + \cos\theta}{\sin\theta} \right) \right]$$

Now, we are in a position to calculate the lift coefficient for a thin, symmetric airfoil.

The total circulation around the airfoil is

$$\Gamma = \int_0^{c} \gamma(\xi) \, d\xi$$

$$\Gamma = \frac{c}{2} \int_0^{\pi} \gamma(\theta) \cdot \sin\theta \, d\theta$$

$$\Gamma = \frac{c}{2} \int_0^{\pi} 2 V_\infty \cdot \frac{(1 + \cos\theta)}{\sin\theta} \cdot \sin\theta \, d\theta$$

$$\Gamma = \alpha \cdot c \, V_\infty \int_0^{\pi} (1 + \cos\theta) \, d\theta$$

$$\Gamma = \alpha \pi c V_\infty$$

Kutta-Joukowski

$$L' = \rho_\infty V_\infty \Gamma = \rho_\infty V_\infty^2 c \alpha \pi$$

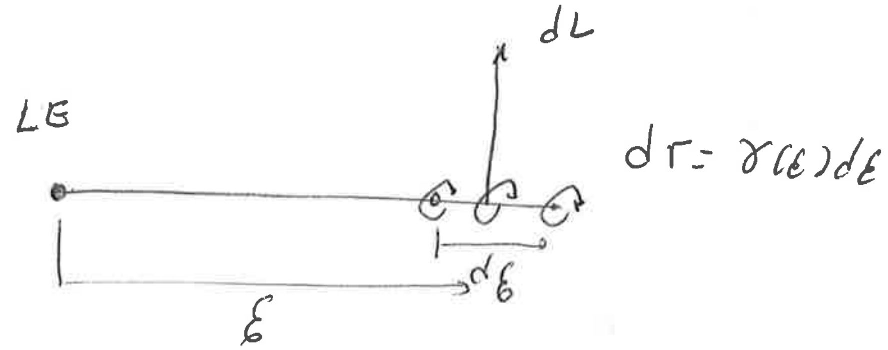
$$C_L = \frac{L'}{q_\infty S} \Rightarrow \quad S = c(1)$$

$$C_L = \frac{\rho_\infty V_\infty^2 . c . \alpha \pi}{\frac{1}{2} \rho_\infty V_\infty^2 . c}$$

$\left[C_L = 2\pi\alpha.\right]$ and $\left[\text{lift slope} = \frac{dC_L}{d\alpha} = 2\pi\right]$

They states the theorietical result that the lift coefficient is linearly proportional to angle of attack.

The moment about the leading edge can be calculated as follows.



$dL = \rho_\infty V_\infty d\Gamma$ this dL creates a moment about the L.E.

$dM = -\xi \, dL$. the total moment about the LE (per unit span)

$$M_{LE} = -\int_0^C \xi \, dL = -\rho_\infty V_\infty \int_0^C \xi \cdot \gamma(\xi) \, d\xi$$

$$M_{LE} = -\rho_\infty V_\infty \int_0^{\pi} \frac{c}{2}(1 + \cos\theta) \cdot \gamma(\theta) \cdot \frac{c}{2} \cdot \sin\theta \, d\theta$$

$$M_{LE} = -\frac{\rho_\infty V_\infty^2}{2} \cdot \frac{\pi \alpha}{2} = -q_\infty C^2 \frac{\pi \alpha}{2}$$

The moment coefficient is

$$C_{m,Le} = \frac{M'_{LE}}{q_\infty S.C}$$

$$C_{m.Le} = \frac{M'_{LE}}{q_\infty \cdot c^2} = -\frac{\pi \cdot \alpha}{2} \quad \text{and } \pi \alpha = \frac{C_L}{2}$$

$$\left[C_{m\,Le} = -\frac{C_L}{4} \right]$$

From before $C_{m\,c/4} = C_{M\,LE} + \frac{C_L}{4}$

$$C_{m\,c/4} = -\frac{C_L}{4} + \frac{C_L}{4} = 0$$

By definition for the center of pressure as that point about which the moment are zero. The theoritical result that the center of pressure is at the quartor-chord point for a symmetric airfoil.

By definition, the point on an airfoil where moments are independent of angle of attack is called the aerodynamic center. For symmetric airfoil, we just obtained the theoretical result that the quarter-chord point is both the center of pressure and aerodynamic center.