The Finite Element Method

Section 4

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Section Objectives

Having demonstrated the finite element method for a 1-D toy problem we now consider a more 'relevant' 2-D problem.

In this section we will:

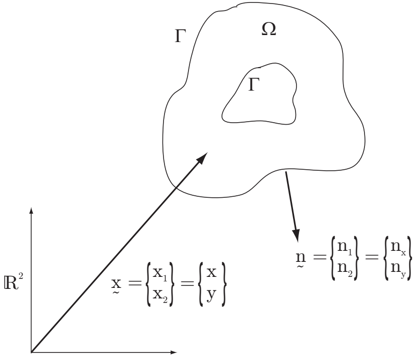
- demonstrate the finite element method for a 2-D scalar field BVP
- consider the specific problem of classical linear heat conduction
- demonstrate (as for the 1-D problem) the process

$$(S) \rightarrow (W) \rightarrow (G) \rightarrow (M)$$

to carry out the approximation

2-D BVP Scalar Field

A domain Ω exists in two-dimensional space, and has a boundary Γ which may have a number of separate locations which need not be contiguous. The unit outward normal vector to the boundary is denoted $\mathbf{n}$.



Mathematical Preliminaries

We require the solution of the scalar variable u across the domain. Differentiation of this scalar variable is denoted

$$u_{,i} = u_{,x_i} = \frac{\partial u}{\partial x_i}$$

Further differentiation is denoted by a repeated suffix, and the summation convention is adopted

$$u_{,ii} = u_{,11} + u_{,22} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

In order to follow our finite element approximation process we need to be able to integrate by parts

Visualizer

Integration by parts using Gauss' Theorem

Classical Linear Heat Conduction: Strong Form

We define the Cartesian components of the heat flux vector q_i, the temperature u, the thermal conductivity κ_{ij}, and the heat supply per unit volume l. The temperature is related to heat flux by means of a constitutive equation, in this case the Fourier law

$$q_i = -\kappa_{ij} u_{,i} \qquad \kappa_{ij} = \kappa_{ji}$$

For an isotropic material $\kappa_{ij}(\mathbf{x}) = \kappa(\mathbf{x})\delta_{ij}$ in which δ is the Kronecker delta. We can now write the strong form of the linear heat conduction equation as

$$(S) \left\{ \begin{array}{rcll} q_{i,i} & = & l & \text{in } \Omega \text{ (Poisson (heat) equation)} \\ u & = & g & \text{on } \Gamma_g \text{ (Prescribed temp: Dirichlet and essential B.C.)} \\ -q_i n_i & = & h & \text{on } \Gamma_h \text{ (Prescribed flux: Neumann and natural B.C.)} \end{array} \right.$$

In which, for example, the Dirichlet boundary condition implies

$$u(\mathbf{x}) = g(\mathbf{x}) \ \forall \ \mathbf{x} \in \Gamma_g$$

Classical Linear Heat Conduction: Weak Form

In a similar fashion to the procedure adopted for the 1-D BVP we now obtain the weak for of the linear heat conduction equation.

Assume a trial solution space $\mathcal{S}$ and a variation space $\mathcal{V}$. First we require the integral of the weighted residual over the domain to be equal to zero

$$\int_\Omega w(q_{i,i} - l) d\Omega = \int_\Omega w q_{i,i} d\Omega - \int_\Omega w l d\Omega = 0$$

Visualizer

Integrate by parts to arrive at the weak form

$$(W) \left\{ \begin{array}{c} \text{find } u \in \mathcal{S} \text{ s.t. } \forall\ w \in \mathcal{V} \\ -\int_\Omega w_{,i} q_i d\Omega = \int_\Omega w l d\Omega + \int_{\Gamma_h} w h d\Gamma \end{array} \right.$$

Symmetric Bilinear Forms

Defining the following symmetric bilinear forms

$$a(w,u) = \int_\Omega w_{,i} \kappa_{ij} u_{,j} d\Omega$$

$$(w,l) = \int_\Omega wl d\Omega$$

$$(w,h)_\Gamma = \int_{\Gamma_h} whd\Gamma$$

we can express the weak for of the linear heat conduction equation succinctly as

$$a(w,u) = (w,l) + (w,h)_\Gamma$$

Symmetric Bilinear Forms: Alternative Notation

The gradient of a scalar field is defined as

$$\nabla u = \{u_{,i}\} = \left\{ \begin{array}{c} u_{,1} \\ u_{,2} \end{array} \right\}$$

In two dimensions the conductivity matrix is

$$\kappa = [\kappa_{ij}] = \left[\begin{array}{cc} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{array} \right]$$

Consequently we can write

$$w_{,i}\kappa_{ij}u_{,j} = (\nabla w)^T \kappa (\nabla u)$$

and therefore the symmetric bilinear expression $a(w, u)$ may be rewritten as

$$a(w, u) = \int_\Omega (\nabla w)^T \kappa (\nabla u) d\Omega$$

Galerkin Formulation

We take $\mathcal{S}^h$ and $\mathcal{V}^h$ to be finite-dimensional approximations to the trial solution space $\mathcal{S}$ and the variation space $\mathcal{V}$ respectively. There are restrictions that must be placed on these finite-dimensional approximations:

- $\mathcal{V}^h$ must vanish (or approximately vanish) on Γ_g
- $\mathcal{S}^h$ must permit the approximate solution to be expressed as $u^h = v^h + g^h$ where $v^h \in \mathcal{V}$ and g^h satisfies (at least approximately) $u = g$ on Γ_g

Substitution of these approximations into the symmetric bilinear weak form of the equation gives

$$a(w^h, v^h + g^h) = (w^h, l) + (w^h, h)_\Gamma$$

This may be rearranged to give the Galerkin form of the equation

$$\text{(G)} \qquad a(w^h, v^h) = (w^h, l) + w^h, h)_\Gamma - a(w^h, g^h)$$

Approximation Functions

We assume that the problem domain is discretized by a mesh for which the set of global node numbers is denoted

$$\eta = \{1, 2, \cdots, n_{np}\}$$

in which n_{np} is the total number of points.

The domain and boundary of an individual element are denoted

$$\Omega^e \quad \text{and} \quad \Gamma^e$$

We also define the set of nodes for which which $u^h = g$ as $\eta_g \subset \eta$. We then define the complement of η_g in η, i.e. the nodes for which u^h must be determined as $\eta - \eta_g$. The number of nodes in the complement is denoted n_{eq} which corresponds to the number of equations that must be solved to attain an approximate solution to the problem.

Approximation Functions

Members of $\mathcal{V}^h$ are approximated by

$$w^h(\mathbf{x}) = \sum_{A \in \eta - \eta_g} N_A(\mathbf{x}) c_A$$

for which we require that $w^h = 0$ if and only if $c_A = 0$.

Similarly

$$v^h(\mathbf{x}) = \sum_{A \in \eta - \eta_g} N_A(\mathbf{x}) d_A$$

in which d_A is the unknown at node A, in this case the temperature. We also need to approximate the boundary condition as

$$g^h(\mathbf{x}) = \sum_{A \in \eta_g} N_A(\mathbf{x}) g_A, \qquad g_A = g(\mathbf{x}_A)$$

Matrix Form

These approximations are then substituted into the Galerkin form of the equation (G).

$$a\left(\sum_{A\in\eta-\eta_g} N_A c_A, \sum_{B\in\eta-\eta_g} N_B d_B\right) = \left(\sum_{A\in\eta-\eta_g} N_A c_A, l\right) + \left(\sum_{A\in\eta-\eta_g} N_A c_A, h\right)_\Gamma - a\left(\sum_{A\in\eta-\eta_g} N_A c_A, \sum_{B\in\eta_g} N_B g_B\right)$$

This may be rearranged to give: for $A \in \eta - \eta_g$

$$a\left(N_A, \sum_{B\in\eta-\eta_g} N_B\right) d_B = (N_A, l) + (N_A, h)_\Gamma - \sum_{B\in\eta_g} a(N_A, N_B) g_B$$

or more concisely

$$\text{(M)} \qquad \mathbf{Kd} = \mathbf{F}$$

Summary

Once again we have followed

$$(S) \to (W) \to (G) \to (M)$$

In particular we have:

- demonstrated the finite element method for a 2-D scalar field BVP, specifically the classical linear heat conduction problem
- shown how to account for non-contiguous boundaries, and non-continuous boundary conditions

The next step is to consider the computational implementation, using an extension of the method proposed for the 1-D problem