$$+\frac{1.0}{y_{1n}-y_m}\left[1.0+\frac{x_m-x_{1n}}{\sqrt{(x_m-x_{1n})^2+(y_m-y_{1n})^2}}\right]$$
$$-\frac{1.0}{y_{2n}-y_m}\left[1.0+\frac{x_m-x_{2n}}{\sqrt{(x_m-x_{2n})^2+(y_m-y_{2n})^2}}\right]\Bigg\} \quad \textbf{(7.44)}$$

Summing the contributions of all the vortices to the downwash at the control point of the mth panel,

$$w_m = \sum_{n=1}^{2N} w_{m,n} \quad \textbf{(7.45)}$$

Let us now apply the tangency requirement defined by equations (7.41) and (7.42). Since we are considering a planar wing in this section, $(dz/dx)_m = 0$ everywhere and $\phi = 0$. The component of the free-stream velocity perpendicular to the wing is $U_\infty \sin\alpha$ at any point on the wing. Thus, the resultant flow will be tangent to the wing if the total vortex-induced downwash at the control point of the mth panel, which is calculated using equation (7.45) balances the normal component of the free-stream velocity:

$$w_m + U_\infty \sin\alpha = 0 \quad \textbf{(7.46)}$$

For small angles of attack,

$$w_m = -U_\infty\alpha \quad \textbf{(7.47)}$$

In Example 7.2, we will solve for the aerodynamic coefficients for a wing that has a relatively simple planform and an uncambered section. The vortex lattice method will be applied using only a single lattice element in the chordwise direction for each spanwise subdivision of the wing. Applying the boundary condition that there is no flow through the wing at only one point in the chordwise direction is reasonable for this flat-plate wing. However, it would not be adequate for a wing with cambered sections or a wing with deflected flaps.

EXAMPLE 7.2: Use the vortex lattice method (VLM) to calculate the aerodynamic coefficients for a swept wing

Let us use the relations developed in this section to calculate the lift coefficient for a swept wing. So that the calculation procedures can be easily followed, let us consider a wing that has a relatively simple geometry (i.e., that illustrated in Fig. 7.31). The wing has an aspect ratio of 5, a taper ratio of unity (i.e., $c_r = c_t$), and an uncambered section (i.e., it is a flat plate). Since the taper ratio is unity, the leading edge, the quarter-chord line, the three-quarter-chord line, and the trailing edge all have the same sweep, 45°. Since

$$AR = 5 = \frac{b^2}{S}$$

and since for a swept, untapered wing

$$S = bc$$

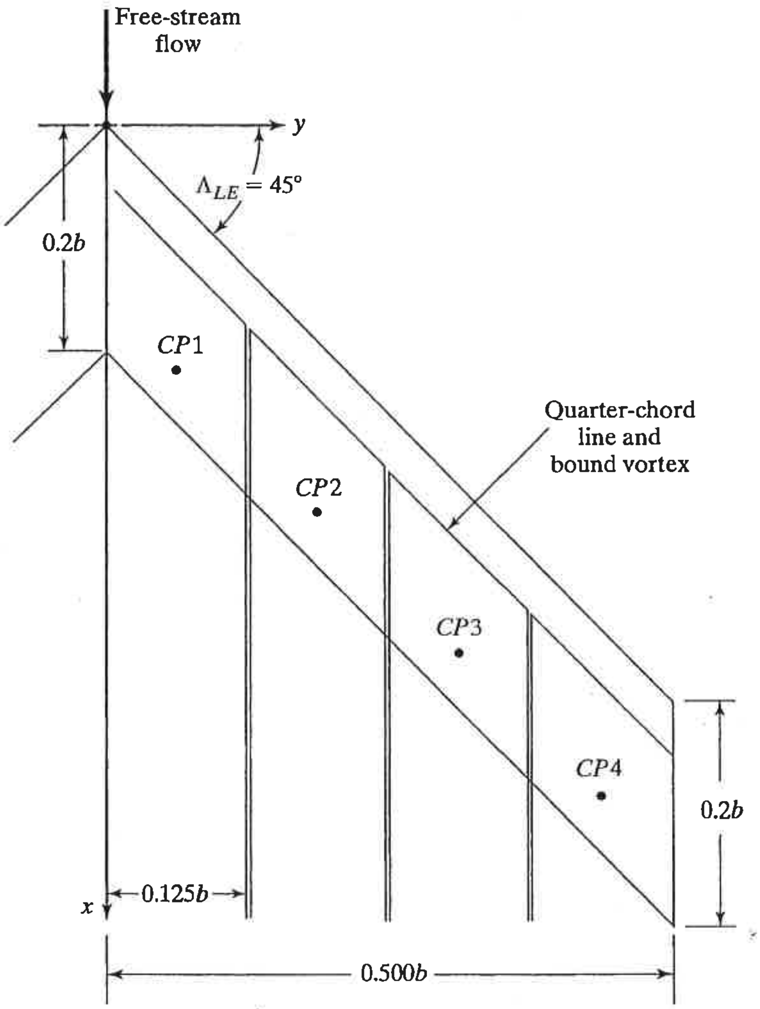

Figure 7.31 Four-panel representation of a swept planar wing, taper ratio of unity, AR = 5, Λ = 45°.

it is clear that $b = 5c$. Using this relation, it is possible to calculate all of the necessary coordinates in terms of the parameter b. Therefore, the solution does not require that we know the physical dimensions of the configuration.

Solution: The flow field under consideration is symmetric with respect to the $y = 0$ plane (xz plane); that is, there is no yaw. Thus, the lift force acting at a point on the starboard wing $(+y)$ is equal to that at the corresponding point on the port wing $(-y)$. Because of symmetry, we need only to solve for the strengths of the vortices of the starboard wing. Furthermore, we need to apply the tangency condition [i.e., equation (7.47)] only at the control points of the starboard wing. However, we must remember to include the contributions of the horseshoe vortices of the port wing to the velocities induced at these control points (of the starboard wing). Thus, for this planar symmetric flow, equation (7.45) becomes

$$w_m = \sum_{n=1}^{N} w_{m,ns} + \sum_{n=1}^{N} w_{m,np}$$

where the symbols s and p represent the starboard and port wings, respectively.

The planform of the starboard wing is divided into four panels, each panel extending from the leading edge to the trailing edge. By limiting ourselves to only four spanwise panels, we can calculate the strength of the horseshoe vortices using only a pocket electronic calculator. Thus, we can more easily see how the terms are to be evaluated. As before, the bound portion of each horseshoe vortex coincides with the quarter-chord line of its panel and the trailing vortices are in the plane of the wing, parallel to the x axis. The control points are designated by the solid symbols in Fig. 7.31. Recall that $(x_m, y_m, 0)$ are the coordinates of a given control point and that $(x_{1n}, y_{1n}, 0)$ and $(x_{2n}, y_{2n}, 0)$ are the coordinates of the "ends" of the bound-vortex filament AB. The coordinates for a 4 × 1 lattice (four spanwise divisions and one chordwise division) for the starboard (right) wing are summarized in Table 7.2.

Using equation (7.44) to calculate the downwash velocity at the CP of panel 1 (of the starboard wing) induced by the horseshoe vortex of panel 1 of the starboard wing,

$$\begin{aligned} w_{1,1s} = \frac{\Gamma_1}{4\pi}\Bigg\{ & \frac{1.0}{(0.1625b)(-0.0625b) - (0.0375b)(0.0625b)} \\ & \left[\frac{(0.1250b)(0.1625b) + (0.1250b)(0.0625b)}{\sqrt{(0.1625b)^2 + (0.0625b)^2}}\right. \\ & \left. - \frac{(0.1250b)(0.0375b) + (0.1250b)(-0.0625b)}{\sqrt{(0.0375b)^2 + (-0.0625b)^2}}\right] \\ & + \frac{1.0}{-0.0625b}\left[1.0 + \frac{0.1625b}{\sqrt{(0.1625b)^2 + (0.0625b)^2}}\right] \\ & - \frac{1.0}{0.0625b}\left[1.0 + \frac{0.0375b}{\sqrt{(0.0375b)^2 + (-0.0625b)^2}}\right]\Bigg\} \\ = {} & \frac{\Gamma_1}{4\pi b}(-16.3533 - 30.9335 - 24.2319) \end{aligned}$$

TABLE 7.2 Coordinates of the Bound Vortices and of the Control Points of the Starboard (Right) Wing

Panel	x_m	y_m	x_{1n}	y_{1n}	x_{2n}	y_{2n}
1	0.2125*b*	0.0625*b*	0.0500*b*	0.0000*b*	0.1750*b*	0.1250*b*
2	0.3375*b*	0.1875*b*	0.1750*b*	0.1250*b*	0.3000*b*	0.2500*b*
3	0.4625*b*	0.3125*b*	0.3000*b*	0.2500*b*	0.4250*b*	0.3750*b*
4	0.5875*b*	0.4375*b*	0.4250*b*	0.3750*b*	0.5500*b*	0.5000*b*

Note that, as one would expect, each of the vortex elements induces a negative (downward) component of velocity at the control point. The student should visualize the flow induced by each segment of the horseshoe vortex to verify that a negative value for each of the components is intuitively correct. In addition, the velocity induced by the vortex trailing from A to ∞ is greatest in magnitude. Adding the components together, we find

$$w_{1,1s} = \frac{\Gamma_1}{4\pi b}(-71.5187)$$

The downwash velocity at the CP of panel 1 (of the starboard wing) induced by the horseshoe vortex of panel 1 of the port wing is

$$\begin{aligned}
w_{1,1p} = \frac{\Gamma_1}{4\pi}\Bigg\{ & \frac{1.0}{(0.0375b)(0.0625b) - (0.1625b)(0.1875b)} \\
& \left[\frac{(-0.1250b)(0.0375b) + (0.1250b)(0.1875b)}{\sqrt{(0.0375b)^2 + (0.1875b)^2}}\right. \\
& \left. - \frac{(-0.1250b)(0.1625b) + (0.1250b)(0.0625b)}{\sqrt{(0.1625b)^2 + (0.0625b)^2}}\right] \\
& + \frac{1.0}{-0.1875b}\left[1.0 + \frac{0.0375b}{\sqrt{(0.0375b)^2 + (0.1875b)^2}}\right] \\
& - \frac{1.0}{-0.0625b}\left[1.0 + \frac{0.1625b}{\sqrt{(0.1625b)^2 + (0.0625b)^2}}\right]\Bigg\} \\
= {} & \frac{\Gamma_1}{4\pi b}[-6.0392 - 6.3793 + 30.9335] \\
= {} & \frac{\Gamma_1}{4\pi b}(18.5150)
\end{aligned}$$

Similarly, using equation (7.44) to calculate the downwash velocity at the CP of panel 2 induced by the horseshoe vortex of panel 4 of the starboard wing, we obtain

$$\begin{aligned}
w_{2,4s} = \frac{\Gamma_4}{4\pi}\Bigg\{ & \frac{1.0}{(-0.0875b)(-0.3125b) - (-0.2125b)(-0.1875b)} \\
& \left[\frac{(0.1250b)(-0.0875b) + (0.1250b)(-0.1875b)}{\sqrt{(-0.0875b)^2 + (-0.1875b)^2}}\right. \\
& \left. - \frac{(0.1250b)(-0.2125b) + (0.1250b)(-0.3125b)}{\sqrt{(-0.2125b)^2 + (-0.3125b)^2}}\right]
\end{aligned}$$

$$+ \frac{1.0}{0.1875b}\left[1.0 + \frac{-0.0875b}{\sqrt{(-0.0875b)^2 + (-0.1875b)^2}}\right]$$

$$- \frac{1.0}{0.3125b}\left[1.0 + \frac{-0.2125b}{\sqrt{(-0.2125b)^2 + (-0.3125b)^2}}\right]\Bigg\}$$

$$= \frac{\Gamma_4}{4\pi b}[-0.60167 + 3.07795 - 1.40061]$$

$$= \frac{\Gamma_4}{4\pi b}(1.0757)$$

Again, the student should visualize the flow induced by each segment to verify that the signs and the relative magnitudes of the components are individually correct.

Evaluating all of the various components (or influence coefficients), we find that at control point 1

$$w_1 = \frac{1}{4\pi b}[(-71.5187\Gamma_1 + 11.2933\Gamma_2 + 1.0757\Gamma_3 + 0.3775\Gamma_4)_s + (+18.5150\Gamma_1 + 2.0504\Gamma_2 + 0.5887\Gamma_3 + 0.2659\Gamma_4)_p]$$

At CP 2,

$$w_2 = \frac{1}{4\pi b}[(+20.2174\Gamma_1 - 71.5187\Gamma_2 + 11.2933\Gamma_3 + 1.0757\Gamma_4)_s + (+3.6144\Gamma_1 + 1.1742\Gamma_2 + 0.4903\Gamma_3 + 0.2503\Gamma_4)_p]$$

At CP 3,

$$w_3 = \frac{1}{4\pi b}[(+3.8792\Gamma_1 + 20.2174\Gamma_2 - 71.5187\Gamma_3 + 11.2933\Gamma_4)_s + (+1.5480\Gamma_1 + 0.7227\Gamma_2 + 0.3776\Gamma_3 + 0.2179\Gamma_4)_p]$$

At CP 4,

$$w_4 = \frac{1}{4\pi b}[(+1.6334\Gamma_1 + 3.8792\Gamma_2 + 20.2174\Gamma_3 - 71.5187\Gamma_4)_s + (+0.8609\Gamma_1 + 0.4834\Gamma_2 + 0.2895\Gamma_3 + 0.1836\Gamma_4)_p]$$

Since it is a planar wing with no dihedral, the no-flow condition of equation (7.47) requires that

$$w_1 = w_2 = w_3 = w_4 = -U_\infty\alpha$$

Thus

$$-53.0037\Gamma_1 + 13.3437\Gamma_2 + 1.6644\Gamma_3 + 0.6434\Gamma_4 = -4\pi bU_\infty\alpha$$
$$+23.8318\Gamma_1 - 70.3445\Gamma_2 + 11.7836\Gamma_3 + 1.3260\Gamma_4 = -4\pi bU_\infty\alpha$$

$$+5.4272\Gamma_1 + 20.9401\Gamma_2 - 71.1411\Gamma_3 + 11.5112\Gamma_4 = -4\pi bU_\infty\alpha$$
$$+2.4943\Gamma_1 + 4.3626\Gamma_2 + 20.5069\Gamma_3 - 71.3351\Gamma_4 = -4\pi bU_\infty\alpha$$

Solving for $\Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4, we find that

$$\Gamma_1 = +0.0273(4\pi bU_\infty\alpha) \quad \textbf{(7.48a)}$$

$$\Gamma_2 = +0.0287(4\pi bU_\infty\alpha) \quad \textbf{(7.48b)}$$

$$\Gamma_3 = +0.0286(4\pi bU_\infty\alpha) \quad \textbf{(7.48c)}$$

$$\Gamma_4 = +0.0250(4\pi bU_\infty\alpha) \quad \textbf{(7.48d)}$$

Having determined the strength of each of the vortices by satisfying the boundary conditions that the flow is tangent to the surface at each of the control points, the lift of the wing may be calculated. For wings that have no dihedral over any portion of the wing, all the lift is generated by the free-stream velocity crossing the spanwise vortex filament, since there are no side-wash or backwash velocities. Furthermore, since the panels extend from the leading edge to the trailing edge, the lift acting on the nth panel is

$$l_n = \rho_\infty U_\infty \Gamma_n \quad \textbf{(7.49)}$$

which is also the lift per unit span. Since the flow is symmetric, the total lift for the wing is

$$L = 2\int_0^{0.5b} \rho_\infty U_\infty \Gamma(y)\, dy \quad \textbf{(7.50a)}$$

or, in terms of the finite-element panels,

$$L = 2\rho_\infty U_\infty \sum_{n=1}^{4} \Gamma_n \Delta y_n \quad \textbf{(7.50b)}$$

Since $\Delta y_n = 0.1250b$ for each panel,

$$L = 2\rho_\infty U_\infty 4\pi bU_\infty\alpha(0.0273 + 0.0287 + 0.0286 + 0.0250)0.1250b$$
$$= \rho_\infty U_\infty^2 b^2 \pi\alpha(0.1096)$$

To calculate the lift coefficient, recall that $S = bc$ and $b = 5c$ for this wing. Therefore,

$$C_L = \frac{L}{q_\infty S} = 1.096\pi\alpha$$

Furthermore,

$$C_{L,\alpha} = \frac{dC_L}{d\alpha} = 3.443 \text{ per radian} = 0.0601 \text{ per degree}$$

Comparing this value $C_{L,\alpha}$ with that for an unswept wing (such as the results presented in Fig. 7.14), it is apparent that an effect of sweepback is the reduction in the lift-curve slope.

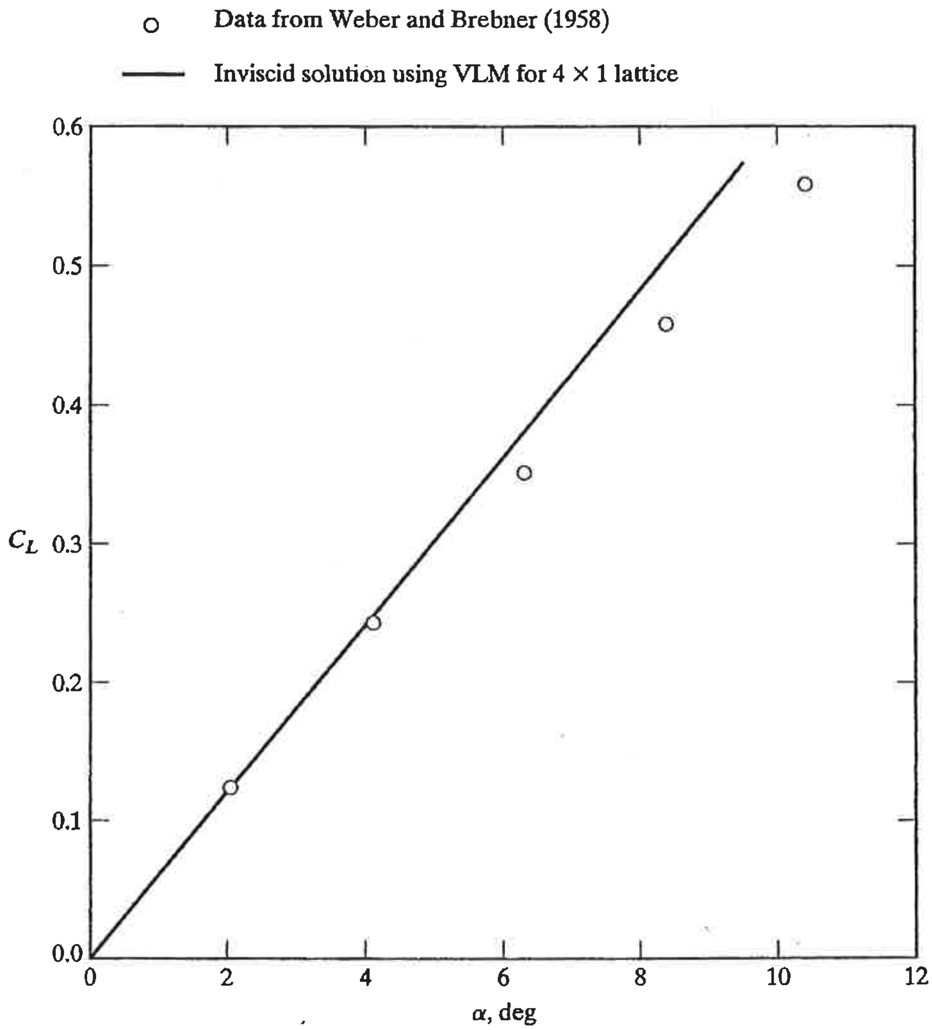

Figure 7.32 Comparison of the theoretical and the experimental lift coefficients for the swept wing of Fig. 7.31 in a subsonic stream.

The theoretical lift curve generated using the VLM is compared in Fig. 7.32 with experimental results reported by Weber and Brebner (1958). The experimentally determined values of the lift coefficient are for a wing of constant chord and of constant section, which was swept 45° and which had an aspect ratio of 5. The theoretical lift coefficients are in good agreement with the experimental values.

Since the lift per unit span is given by equation (7.49), the section lift coefficient for the nth panel is

$$C_{l(n\text{th})} = \frac{l_n}{\frac{1}{2}\rho_\infty U_\infty^2 c_{\text{av}}} = \frac{2\Gamma}{U_\infty c_{\text{av}}} \qquad \textbf{(7.51)}$$

When the panels extend from the leading edge to the trailing edge, such as is the case for the 4 × 1 lattice shown in Fig. 7.31, the value of Γ given in equation (7.48) is used