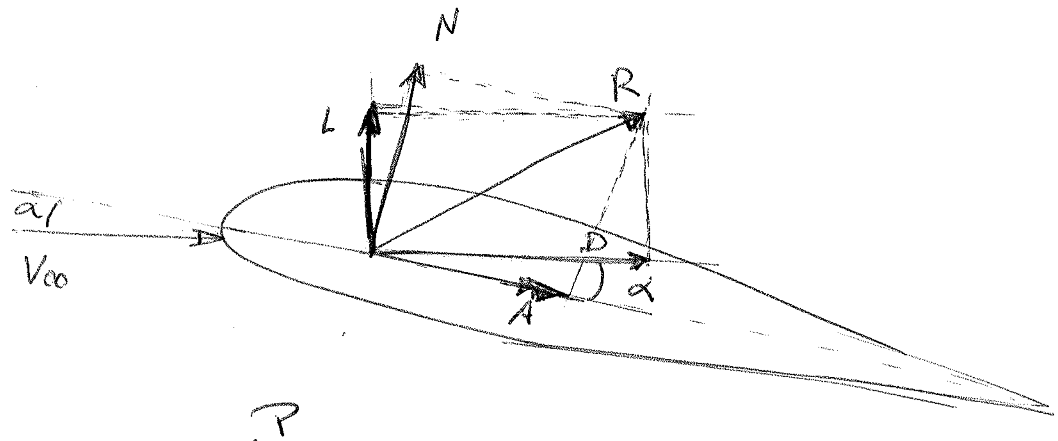
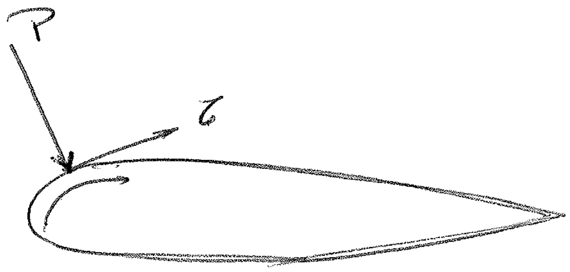
Aerodynamic forces and Moments





$P = P(s) =$ surface pressure distribution

$\tau = \tau(s) =$ surface sheer stress distribution

\+ $L =$ Lift $\equiv$ Component of R. perpendicular to V_∞

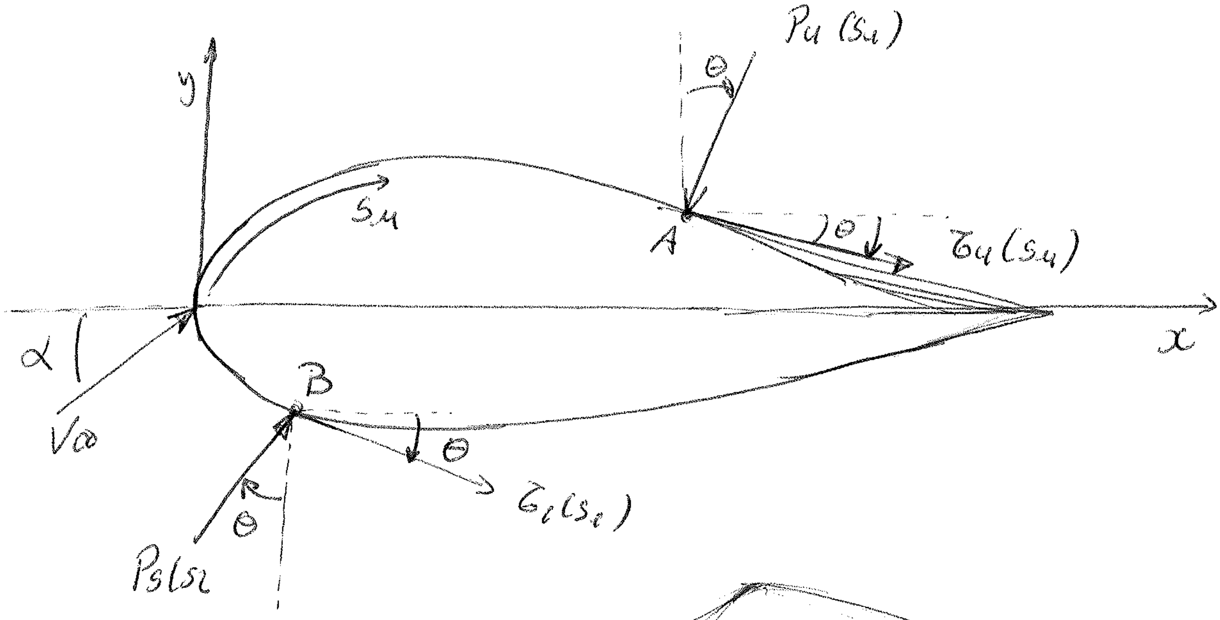
$D =$ Drag $\equiv$ component of R parallel to V_∞

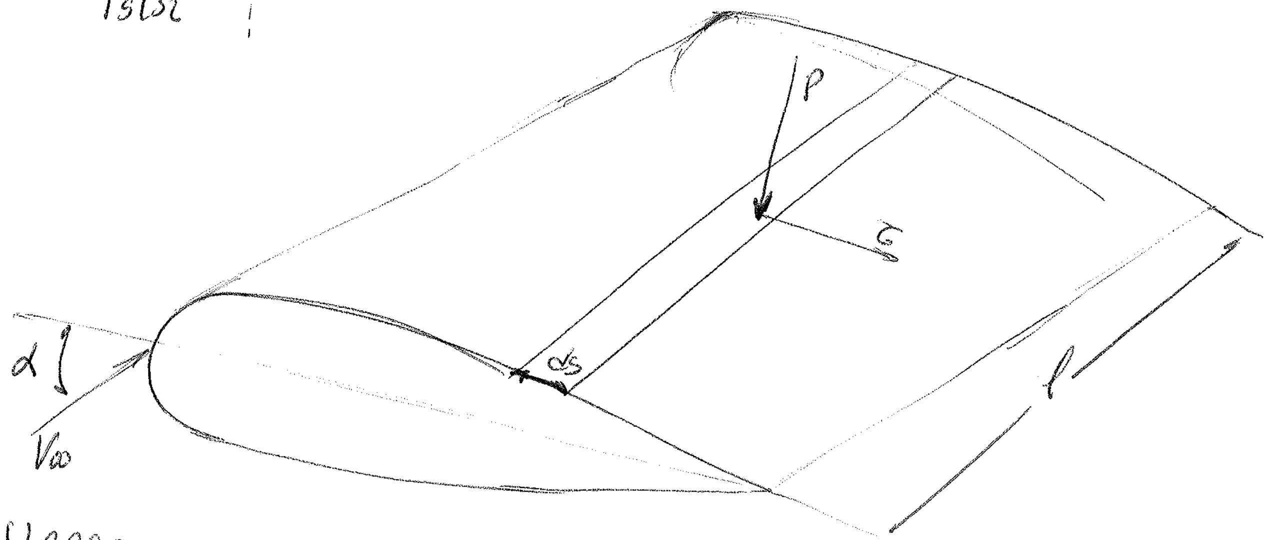
\+ $N =$ normal force $\equiv$ component of R perpendicuar to c

$A =$ Axial force $=$ component of R parallel to c

$$L = N\cos\alpha - A\sin\alpha$$

$$D = N\sin\alpha + A\cos\alpha$$





<u>Upper</u>

$$dN'_u = -P_u\, ds_u \cos\theta - \tau_u\, ds_u \sin\theta$$

$$dA'_u = -P_u\, ds_u \cdot \sin\theta + \tau_u\, ds_u \cos\theta$$

Lower

$$dN'_l = P_l\, ds_l \cos\theta - \tau_l\, ds_u \sin\theta$$

$$dA'_u = P_L\, ds_l \cos\theta + \tau_l\, ds_l \cos\theta$$

The total normal and axial forces per unit span

$$N' = -\int_{LE}^{TE} (P_u \cos\theta + \tau_u \sin\theta)\, ds_u + \int_{LE}^{TE} (P_L \cos\theta - \tau_L \sin\theta)\, ds_L$$

$$A' = \int_{LE}^{TE} (-P_u \sin\theta + \tau_u \cos\theta)\, ds_u + \int_{LE}^{TE} (P_L \sin\theta - \tau_L \cos\theta)\, ds_L$$

L and D are obtained replacing N' and A'.

The aerodynamic moment exerted on the body depends on the point about which moments are taken.

Convention moments.

Increase α (pitch up)
If it is positive (pitch down)
decrease α

Upper

$$dM'_u = (P_u \cos\theta + \tau_u \sin\theta)\, x\, ds_u + (-P_u \sin\theta + \tau_u \cos\theta)\, y\, ds_u$$

Lower

$$dM'_L = (-P_L \cos\theta + \tau_L \sin\theta)\, x\, ds_L + (P_L \sin\theta + \tau_L \cos\theta)\, y\, ds_L$$

$$M'_{LE} = \int_{LE}^{TE} [(P_u \cos\theta + \tau_u \sin\theta)x - (P_u \sin\theta - \tau_u \cos\theta)y]\, ds_u$$

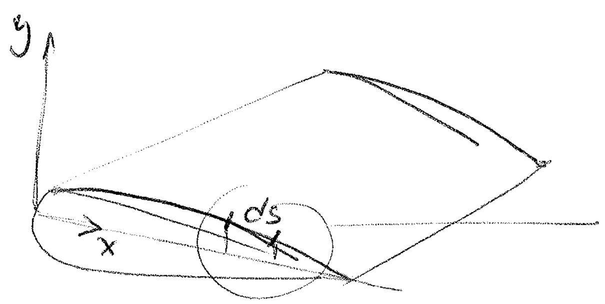
$$+ \int_{LE}^{TE} [(-P_l \cos\theta + \tau_L \sin\theta)x + (P_L \sin\theta + \tau_L \cos\theta\, y)]\, ds_L$$

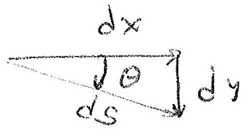
The equations and θ, x, and y are known functions of s for a given body shape. Hence, if P_u, P_L, τ_u and τ_L are known as functions of s.

The sources of the aerodynamics lift, drag and moments on a body are pressure and shear stress distributions integrated over the body.

A major goal of theoritical aerodynamics is to calculate $P(s)$ and $\tau(s)$ for a given body shape and freestream conditions, thus yielding the aerodynamic forces and moments.

- Geometrical relationship of differential lengths





$dx = ds \cos\theta$

$dy = -(ds \sin\theta)$

$S = c(s)$

$$C_n = \frac{1}{c}\int_0^c (C_{p_L} - C_{p_u})\,dx + \int_0^c \left(C_{f_u}\frac{dy_u}{dx} + C_{f,L}\frac{dy_L}{dx}\right)dx$$

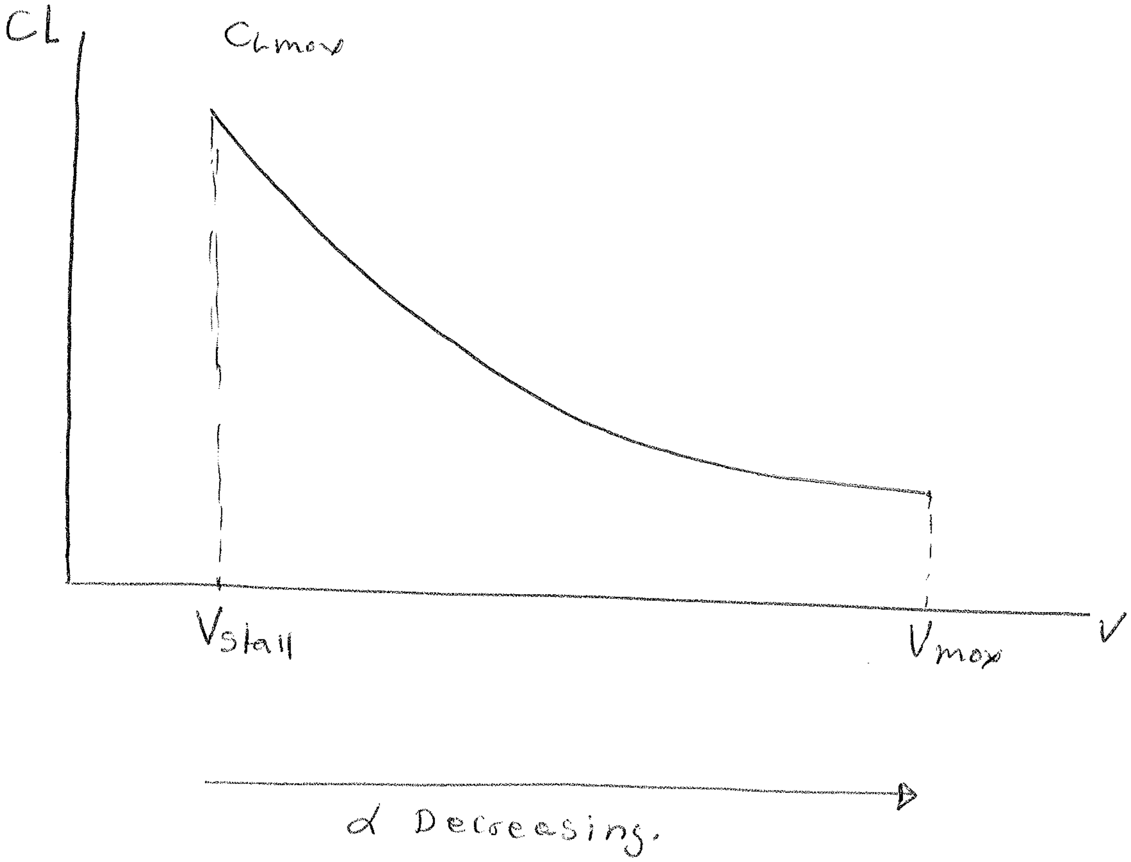
$$C_a = \frac{1}{c}\int_0^c \left(C_{p_u}\frac{dy_u}{dx} - C_{p_L}\frac{dy_L}{dx}\right)dx + \int_0^c (C_{f_u} + C_{f_L})\,dx$$

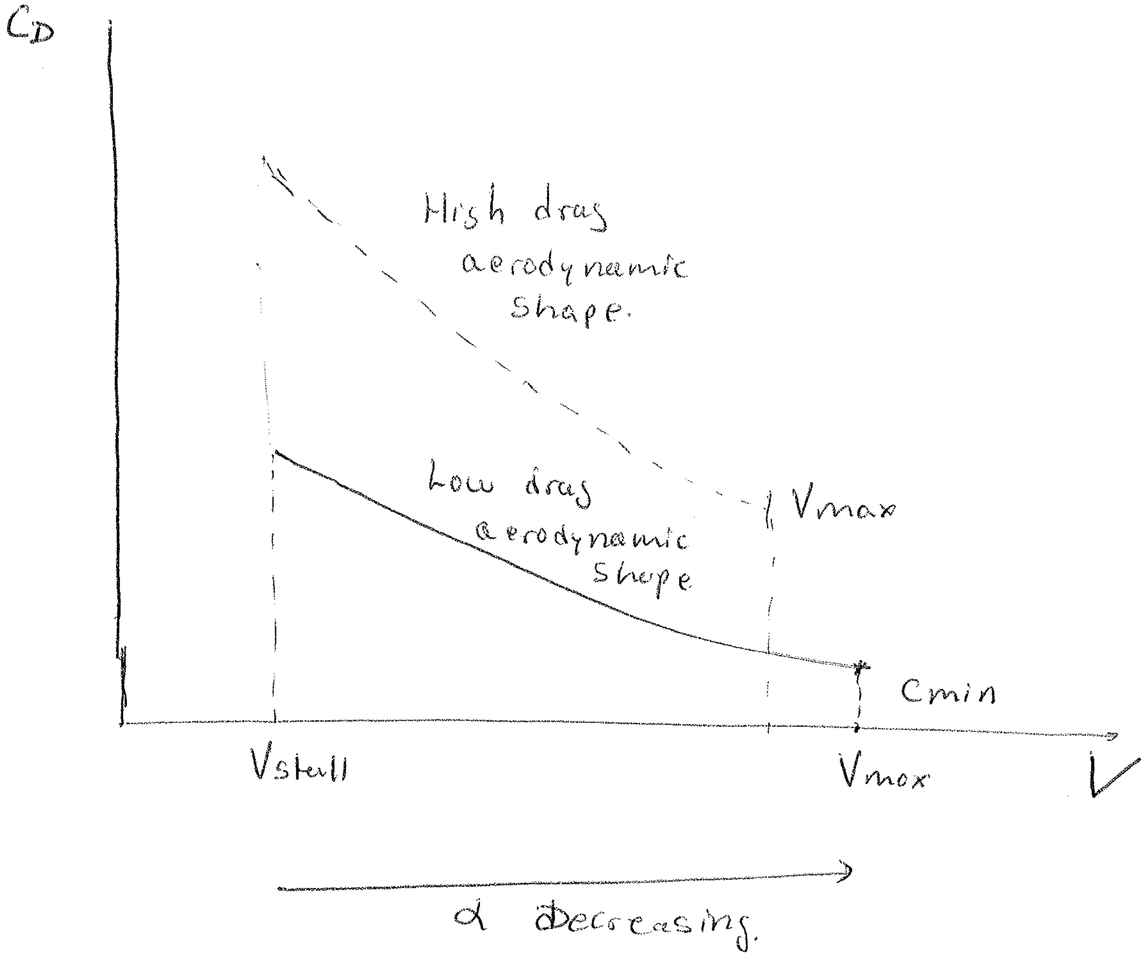
$$C_{m_{LE}} = \frac{1}{c^2}\left[\int_0^c (C_{p_u} - C_{p_L})\,x\,dx - \int_0^c \left(C_{f_u}\frac{dy_u}{dx} + C_{f_L}\frac{dy_L}{dx}\right)x\,dx\right.$$

$$\left. + \int_0^c \left(C_{p_u}\frac{dy_u}{dx} + C_{f_u}\right)y_u\,dx + \int_0^c \left(-C_p\frac{dy_L}{dx} + C_{f_L}\right)y_L\,dx\right]$$

$$C_L = C_n\cos\alpha - C_a\sin\alpha$$

$$C_d = C_n\sin\alpha + C_a\cos\alpha$$


CL
CLmax
Vstall
Vmax
V
α Decreasing.

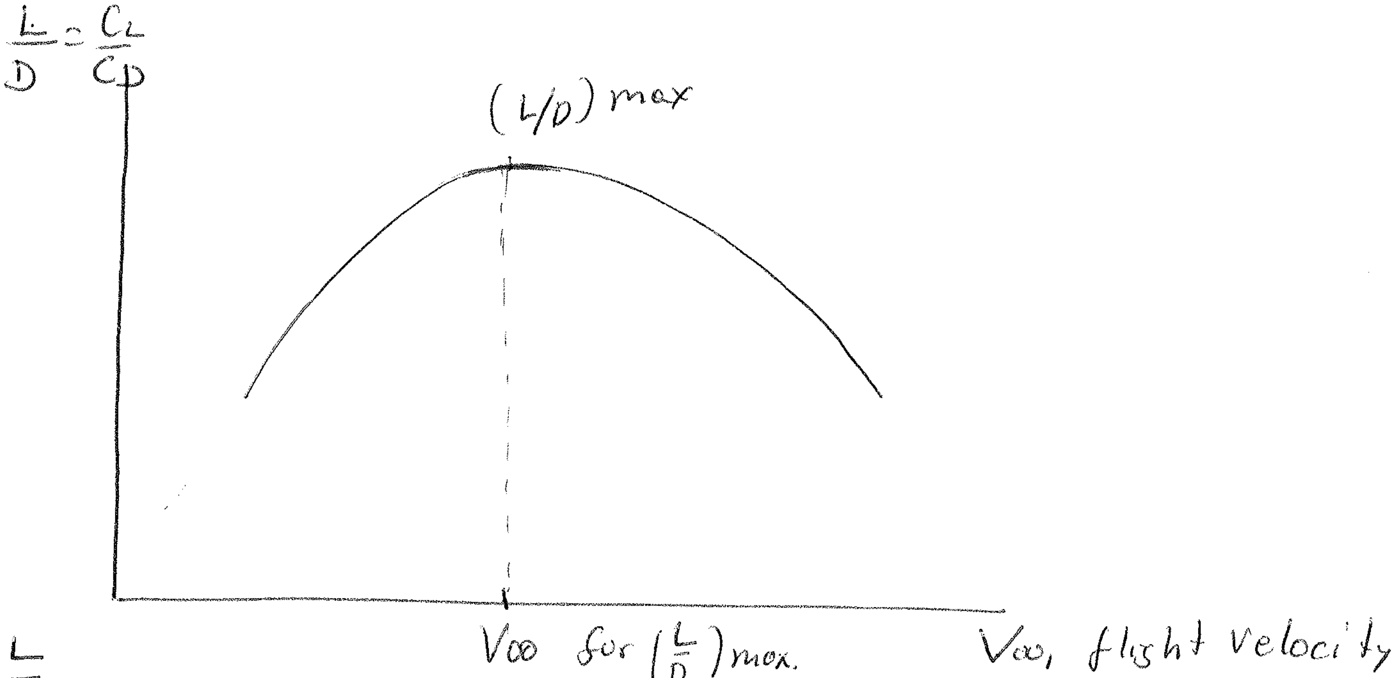

CD
High drag
aerodynamic
shape.
Low drag
aerodynamic
shape
Vmax
Cmin
Vstall
Vmax
V
α Decreasing.

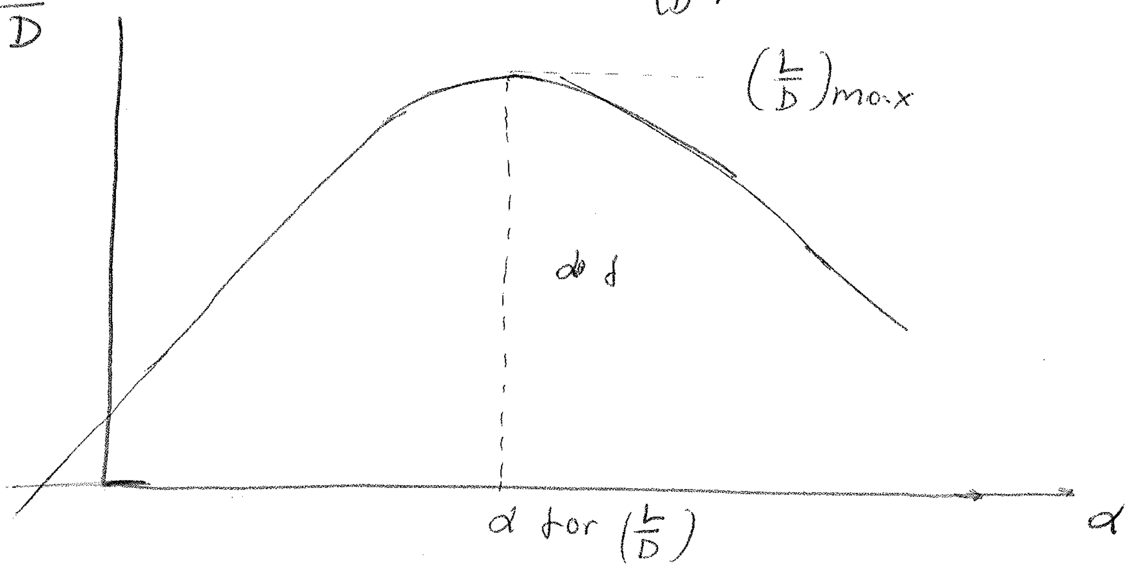
$$\frac{L}{D} = \frac{q_\infty S C_L}{q_\infty S C_D} = \frac{C_L}{C_D}$$

C_L for flying at a given velocity and altitude is determined by airplane's weight and wing area $\left(\frac{W}{S}\right)$ wind loading

The relation $C_L = \frac{L}{q_\infty S} = \frac{W}{q_\infty S}$, so the value of $\frac{L}{D}$ at this velocity is controlled by C_D.

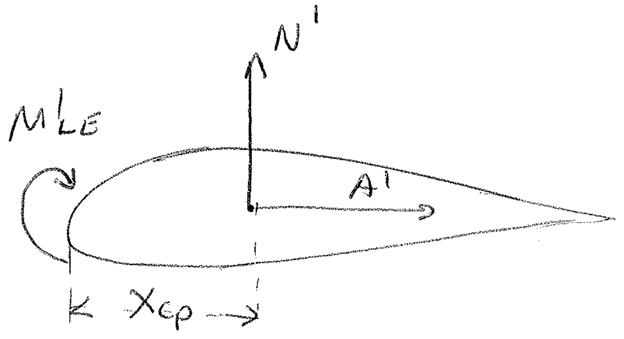
For any velocity, we need to get $\frac{L}{D}$ to be as high as possible. The higher is $\frac{L}{D}$ the more aerodynamically efficient is the body.





Center of Pressure

Equations for Normal and axial forces on the body are due to the distributed Loads impose by pressure and shear stress distribution. These loads generate a moment about the Leading edge as given by the equation of monment.



(+) Pitching moment

N' creates a negative moment about the LE. (Pitching down)

$$M'_{LE} = -(X_{cp})N'$$

$$X_{cp} = -\frac{M'_{LE}}{N'}$$

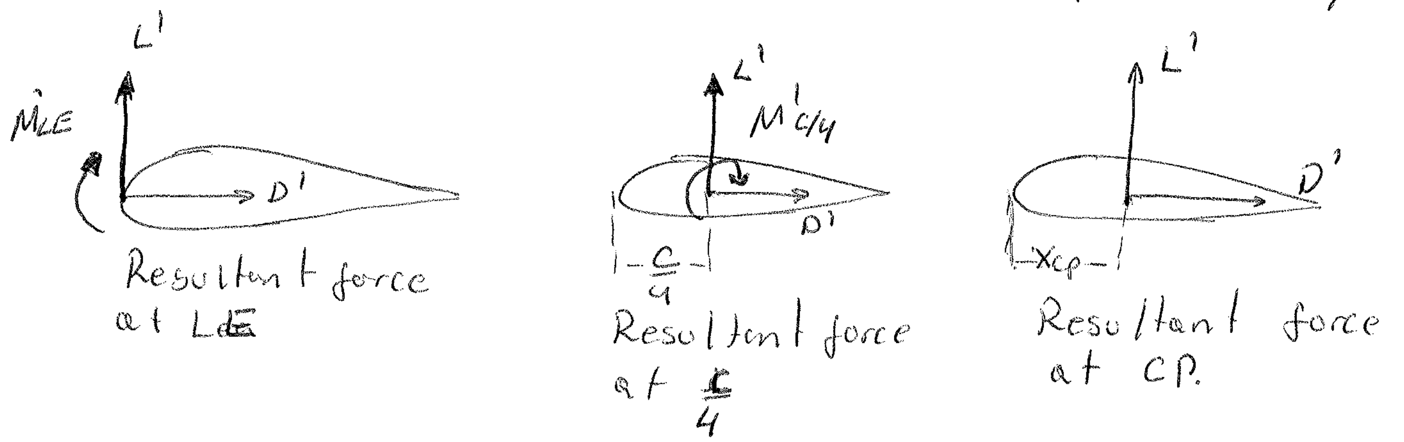
(It is defined as the Cp. It is The location where the resultant of a distributed load effectively acts on the body.)

If the moments were taken about the C.P., the integrated effect of the distributed loads would be zero. Anothe definition is that point on the body about which aerodynamic moments is zero.

If the case ~~wh~~ where the angle of attack of the body is small $\sin\alpha \cong 0$ and $\cos\alpha \cong 1$ then

$L' = D'$ thus $X_{cp} = -\dfrac{M'_{LE}}{L'}$ X_{cp} increases when L' and N' decreases.

If the forces approach zero, the Cp moves to infinity. (No always is a convinient concept in aerodynamics)



$$\left[M'_{LE} = -\frac{c}{4} L' + M'_{c/4} = -X_{cp} L' \right]$$

Solution problem 3.

$$\frac{c}{4} L' = M'_{c/4} - M'_{LE} = -4850 - (-14550)$$
$$= 9700 \ N.m/m$$

At this airfoil location on the wing $\frac{c}{4} = \frac{4.6}{4} = 1.15m$

$$L' = \frac{9700}{1.15} = 8434.8 \ N/m$$

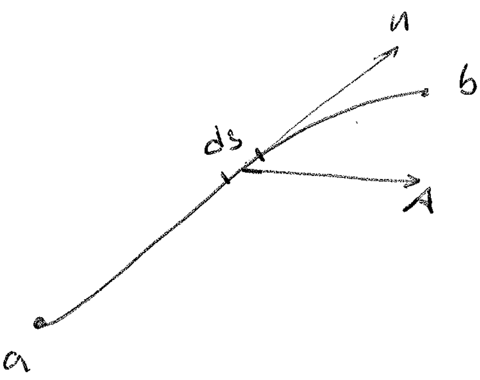
$$-X_{cp} L' = M'_{LE} \qquad X_{cp} = \frac{-M'_{LE}}{L'} = -\frac{(-14550)}{8434.8} = 1.725 \ m$$

Review Algebra

Consider a vector field

$$A = A(x, y, z) = A(r, \theta, z) = A(r, \theta, \phi)$$

Also consider a curve C in space connecting two points a and b

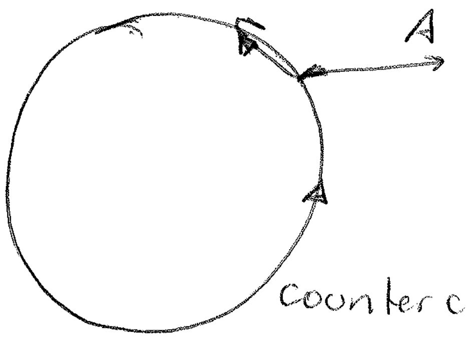


$\vec{n}$ = unit vector tangential

ds = elemental Length

$\mathbf{ds} = n\,ds$ vector

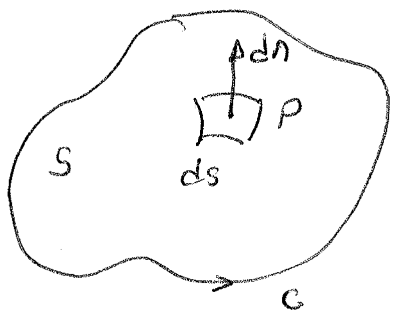
$$\int_{a\;C}^{b} A \cdot ds$$



counterclockwise (+)

$$\oint_{C} A \cdot ds \qquad \text{Closed C.}$$

Surface integrals



S: Surface

C = Closed curve

P = point

ds = elemental area of S

n = normal vector (unit)

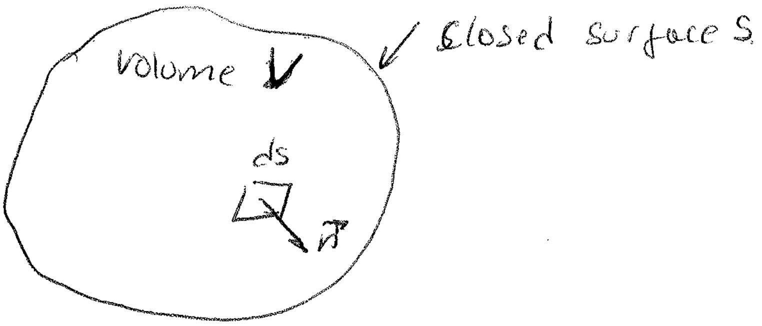
We define a vector elemental area $\mathbf{ds} = \mathbf{n}\,ds$

$$\iint_S P\,\mathbf{ds} =$$ surface integral of a scalar P over the open surface S. (Result is a vector)

$$\iint_S \mathbf{A}\cdot\mathbf{ds} =$$ Surface integral of a vector A over the open surface S (Result is a Scalar)

$$\iint_S \mathbf{A}\times\mathbf{ds} =$$ Surface integral of a vector A over the open surface S (Result is a vector)

If the surface is closed



$$\oiint P\,\mathbf{ds} \qquad \oiint \mathbf{A}\cdot\mathbf{ds} \qquad \oiint \mathbf{A}\times\mathbf{ds}$$

Volume Integral

Consider a Volume V in space. Let ρ be a scalar field in this space. the Volume integral is

$$\iiint_V \rho\,dV =$$ Volume integral of a scalar ρ over the volume V (Scalar)

Let A be a vector field in space. The volume integral over the volume V of the quantity A

$\oiiint A.dV =$ Volome integral of a vector A over the volume V (Resul is a vector)

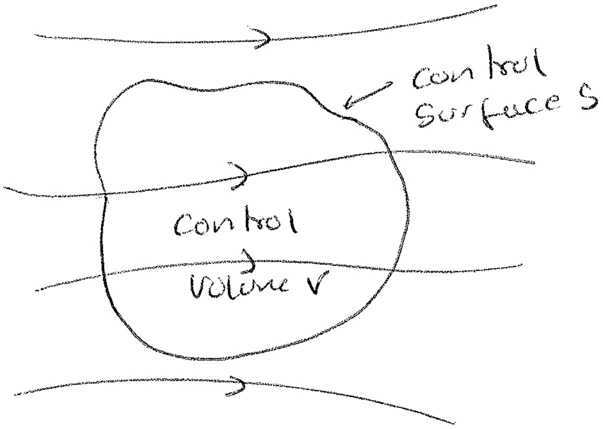
Relation Between Line, Surface and Volume integrals

$\oint_C A.ds = \iint_S (\nabla \times A)\, ds \Rightarrow$ Let A be a vector field. The line integral of A over C is related to surface integral of A over S by Stoke's theorem.

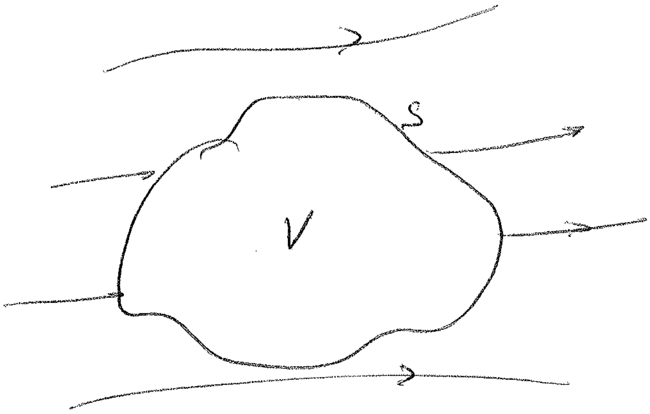
$\oiint_S A.ds = \oiiint_V (\nabla . A) dV \Rightarrow$ Volume V enclosed by the closed surface S. as shown in the above figure. The surface and volume integrals of the vector field A, are related through the divergence theorem.

$\oiint_S P.ds = \oiiint_V \nabla . P dV \Rightarrow$ If P represents a scalar field a vector relationship analogous to the above equation. is given by the gradient theorem.

Finite control Volume Approach

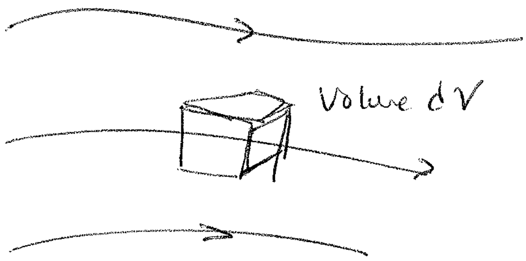


Finite control volume fixed in space with the fluid moving through it.

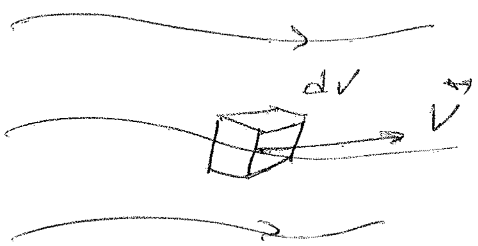


Finite control volume moving with the fluid such that the same fluid particles are always in the same control volume

Infinitesimal Fluid Element Approach



Infinitesimal fluid element fixed in space with fluid moving through it

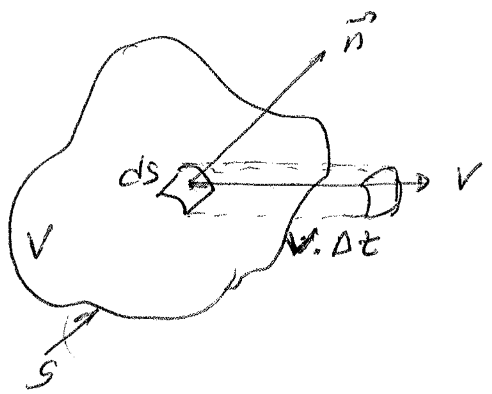


Infinitesimal fluid element moving along a streamline with the velocity V equal to the local flow velocity at each point

Physical Meaning of the Divergence of Velocity

The divergence $\nabla . V$ is physically the timerate of change of the volume of a moving fluid element of fixed mass per unit volume of that element.

Consider a control volume moving with the fluid. This C.V. is always made up of the same fluid particles as it moves with the flow. V and control surface S are changing with time as it moves to different regions of the flow where different values of ρ exist.



Moving control volume used for the physical interpretation of the divergence of velocity at some instant time.

The change in the volume of the control volume ΔV due to just the movement of ds over a time elemental Δt. It is equal to the volume of the long, thin cylinder with base area ds and Altitude $(V \Delta t).n$; that is

$$\Delta V = [(\vec{V}.\Delta t).n]\, ds$$

$$\Delta V = (\vec{V} \Delta t)\, \mathbf{ds} \rightarrow \text{vector.}$$

Over the time increment Δt, the local change in volume of the whole control volume is equal to the summation of the above equation over the total control surface..

$$\oiint_S (\vec{V}.\Delta t).\, ds$$

If this integral is divided by Δt. the result is physically the time rate change of the control volume.

$$\frac{DV}{Dt} = \frac{1}{\Delta t} \oiint_S (\vec{V}\Delta t).\, ds = \oiint \vec{V}.\, ds$$ Applying the divergence theorem.

$$\left[\frac{DV}{Dt} = \oiiint_V (\nabla.\vec{V}).\, dV \right]$$

Think about if the control volume is very small δV.

$$\frac{D\delta V}{Dt} = \oiiint_{\delta V} (\nabla.\vec{V})\, dv$$

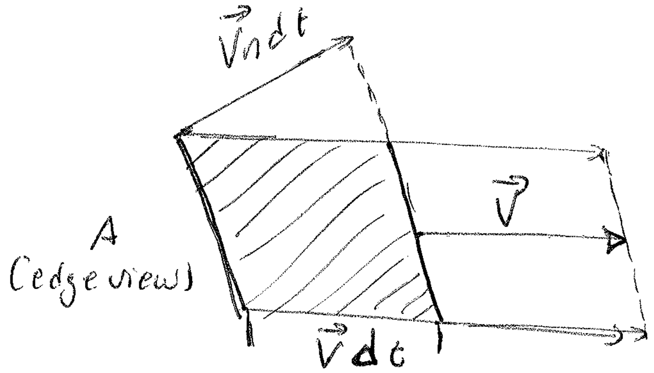
$\nabla.\vec{V}$ is essentially the same value throughout δV. thus the integral is approached to $\nabla.\vec{V}.\,\delta V$

$$\frac{D\delta V}{Dt} = (\nabla.\vec{V})\,\delta V$$

$$\left[\nabla.\vec{V} = \frac{1}{\delta V} \cdot \frac{D\delta V}{Dt} \right]$$

Continuity Equation

Consider a given area A arbitrary oriented in a flow field. We are looking at an edge view of Area A. Let A be small enough such that the flow velocity V is uniform across A.



Volume $= (V_n dt) A$

The mass inside the shaded ~~area~~ volume is

Mass $= \rho . (V_n dt) . A$

$$\dot{m} = \frac{\rho (V_n dt) A}{dt}$$

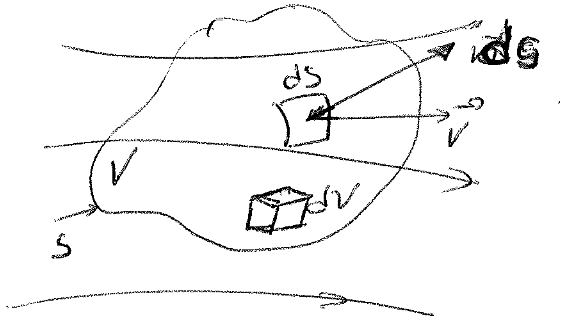
$$\dot{m} = \rho V_n A$$

The mass flux, is defined as the mass flow per unit area.

$$\left[\text{Mass flux} = \frac{\dot{m}}{A} = \rho V_n \right]$$

Physical principle

Consider a flow field wherein all properties vary with spatial location and time. $\rho = (x, y, z, t)$



$$\left\{ \begin{array}{l} \text{The mass flow} \\ \text{out control} \\ \text{volume through} \\ \text{Surface S} \end{array} \right\} = \left\{ \begin{array}{l} \text{Time rate} \\ \text{of decrease} \\ \text{of mass} \\ \text{inside control} \\ \text{volume V.} \end{array} \right\}$$

$$B = C$$

Mass can be neither created nor destroyed.

$$\rho \vec{V}_n \, ds = \rho \vec{V} \cdot ds$$

Elemental mass flow across the area ds is, then

$$B = \oiint_S \rho \vec{V} \cdot ds$$

the mass contained within the elemental volume dv is

$\rho \, dv$. total mass is

$$\oiiint_V \rho \, dv$$

the time rate of increase of mass inside V is then

$$\frac{\partial}{\partial t} \oiiint_V \rho \, dv$$

The time rate of decrease of mass inside V is then

$$-\frac{\partial}{\partial t} \oiiint_V \rho \, dv = C$$

as B must be equal to C

$$\oiint_S \rho \vec{V} \cdot ds = -\frac{\partial}{\partial t} \oiiint_V \rho \, dv$$

$$\left[\frac{\partial}{\partial t} \oiiint_V \rho \, dv + \oiint \rho \vec{V} \cdot ds = 0 \right]$$

Conservation of mass to a finite control Volume, fixed in space.

$$\oiiint_V \frac{\partial \rho}{\partial t}\, dV + \oiint_S \rho \vec{V} \cdot ds = 0$$

Applaying the divergence theorem.

$$\oiint_S (\rho \vec{V}) ds = \oiiint \nabla \cdot (\rho \vec{V}) dV$$

$$\oiiint_V \frac{\partial \rho}{\partial t}\, dv + \oiiint_V \nabla \cdot (\rho \vec{V}) \cdot dv = 0$$

$$\oiiint_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) \right) dv = 0$$

$$\left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0 \right]$$

For incompressible flow and ~~un~~ steady flow.

$$\rho (\vec{\nabla} \cdot \vec{V}) = 0 \qquad \rho \neq 0$$

$$\nabla \cdot \vec{V} = 0 \Rightarrow \frac{\partial \vec{u}}{\partial x} + \frac{\partial \vec{v}}{\partial y} + \frac{\partial \vec{w}}{\partial z} = 0$$

Momentum efuation

Newton's second law is frequently written as

$$F = m\vec{a}$$

General form $F = \frac{d}{dt}(m\vec{V})$. for mass constant is reduced to above efuation $m\vec{V}$. is the momentum of a body of mass m.

Physical principle. Force = time rate of change of momentum.

F: is the force exerted on the fluid as it flows through the control volume) This force comes from two sources

1) Body Forces; gravity. electromagnetic force
2.) Surface forces; Pressure and shear stress acting on the control surfaces.

Conside (f) represents the body force per unit mass exerted on the fluid inside V. the body force on elemental Volume dv

$$[\rho f dv]$$

Total force

$$\text{Body force} = \iiint_V \rho f \, dV$$

The elemental surface force due to pressure acting on elemental of area ds is

$$[-P\,ds]$$

where negative sign indicates that the force is in direction opposite of ds.

$$\text{Pressure force} = -\oiint_S P\,ds$$

In a viscous flow, shear and normal viscous stresses also exert a surface force, $F_{viscous}$.

$$F = \oiiint \rho f\,dv + F_{viscous} - \oiint_S P\,ds$$

Consider the right side of the equation of Newton's second law. The time rate of change of momentum of fluid as it sweeps through the fixed volume is the sum of two terms.

Net flow of momentum out of control volume across surface S $\equiv G$

Time rate of change of momentum due to unsteady fluctuations of flow properties inside V $\equiv \dot{H}$

$$G = \oint\!\!\oint_S (\rho \vec{V} \cdot ds) \cdot \vec{V}$$

The flow has a certain momentum as it enters the C.V. and it has a different momentum as it leaves the C.V. (due in part to the Force)

$\rho \vec{V} \cdot ds$ = mass of fluid.

The momentum of the fluid in the elemental volume dV is

$$(\rho \, dV) \cdot \vec{V}$$

The momentum at any instant is

$$\oint\!\!\oint\!\!\oint_V \rho \vec{V} \, dV$$

The time rate of change due to unsteady flow fluctuations is

$$H = \frac{\partial}{\partial t} \oint\!\!\oint\!\!\oint_V (\rho \vec{V}) \, dV$$

$$\frac{d}{dt}(m\vec{V}) = G + H \Rightarrow \frac{\partial}{\partial t} \oint\!\!\oint\!\!\oint (\rho \vec{V}) dV + \oint\!\!\oint_S (\rho \vec{V}) ds \cdot \vec{V}$$

$$\frac{\partial}{\partial t} \oint\!\!\oint\!\!\oint \rho \vec{V} \cdot dV + \oint\!\!\oint_S (\rho \vec{V}) ds \cdot \vec{V} = -\oint\!\!\oint_S P \, ds + \oint\!\!\oint\!\!\oint \rho \cdot f \cdot ds + F_{viscous}$$

$$-\oiint_S P\,ds = -\oiiint_V \nabla . P\,dv$$

$$\oiiint_V \frac{\partial(\rho\vec{V})}{\partial t}dv + \oiint_S (\rho V . ds)V = -\oiiint_V \nabla . P\,dv + \oiiint_V \rho f\,dv + F_{viscous}$$

$$\oiiint_V \frac{\partial(\rho\vec{V})}{\partial t}dv + \oiiint_V \vec{V}.\nabla(\rho\vec{V})dv = -\oiiint_V \nabla P\,dv + \oiiint_V \rho f\,dv + F_{visco}$$

$$\oiiint_V \frac{\partial \rho \vec{V}}{\partial t} + \vec{\nabla}.(\rho\vec{V}.\vec{V}) + \nabla P - \rho f + F_{viscous} = 0$$

$$\frac{\partial(\rho\vec{V})}{\partial t} + \nabla(\rho\vec{V}.\vec{V}) = -\nabla P + \rho\vec{g} + \cancel{\mu} F_{viscous}$$

$$\boxed{\frac{D(\rho\vec{V})}{Dt} = -\nabla P + \rho g + \mu\nabla^2\vec{V}}$$

x- direction

$$\frac{\partial \rho u}{\partial t} + \nabla.(\rho u . \vec{V}) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

y- direction

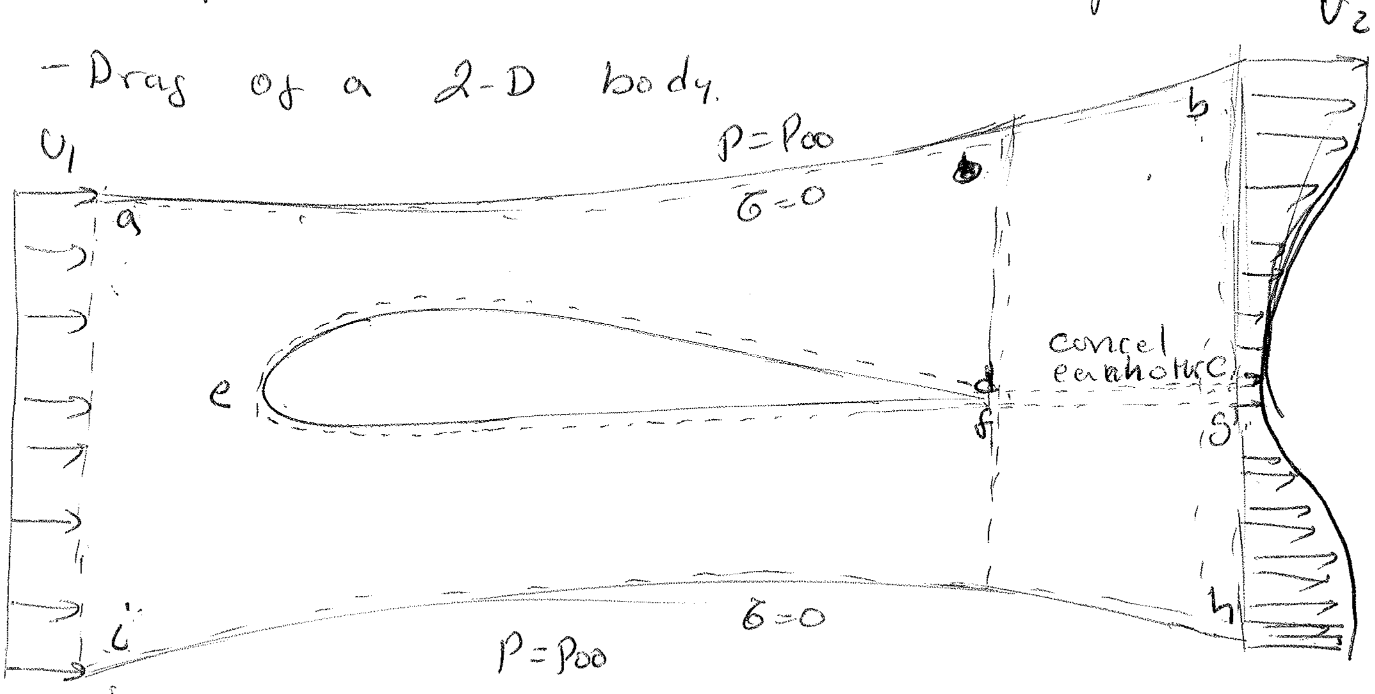
$$\frac{\partial \rho v}{\partial t} + \nabla(\rho v.\vec{V}) = -\frac{\partial P}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

z- direction

$$\frac{\partial \rho w}{\partial t} + \nabla.(\rho w.\vec{V}) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

An application of the momentum equation.

- Drag of a 2-D body.



U_1 is uniform flow

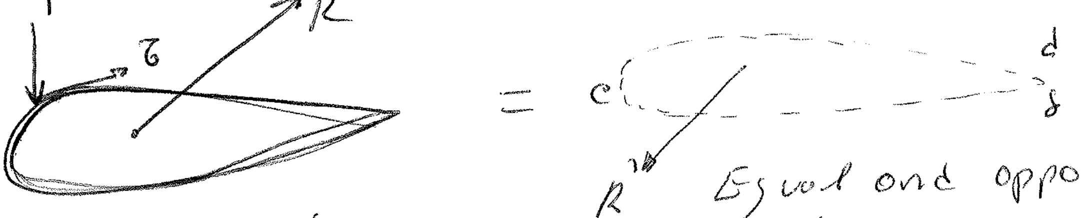
U_2 is a function of $y \Rightarrow U_2 = f(y)$.

1. The pressure distribution over the surface abhi

$$-\iint_{abhi} P.\,ds$$

2. The surface force on def created by the presence of the body.

$$\text{Surface force} = -\iint_{abhi} P\,ds - R'$$



Flow exerts P and τ on the surface of the body. giving a resultant force R

Equal and opposite reaction, body exerts a surface force on the ~~action~~ section of the C.V. edf

$$\frac{\partial}{\partial t}\iiint \rho \vec{V} dV + \oiint_S (\rho \vec{V}\cdot ds)\vec{V} = -\iint_{abhi} P ds - R'$$

Assuming steady flow.

$$R' = -\oiint_S (\rho \vec{V}\cdot ds)\vec{V} + \iint_{abhi} P ds$$

This is a vector equation. Consider again the control volume. Taking the x-component, noting that the inflow and outflow velocities u_1 and u_2, the component of R' is the aerodynamic drag per unit span D'.

$$D' = -\oiint_S (\rho \vec{V}\cdot ds)u - \iint_{abhi} (P.ds)_x$$

$\int(P.ds)_x$ is the x component of the pressure force exerted on the elemental area ds of the control surface. The boundaries of the C.V. abhi are chosen far away from the body such that P is constant along these boundaris. For constant pressure.

$$\iint_{abhi} (P.ds)_x = 0$$

Then $$D' = -\oiint_S (\rho \vec{V}.ds)u$$

1. The section ab, hi and def are streamlines of flow. Since by definition V is parallel to streamline and ds is perpendicular to the control surface, along these sections V and ds are perpendicular vector and hence $V \cdot ds = 0$ As a result the contribution of ab, hi and def to the integral is zero.

2. The cuts cd an fg are adjacent to each other. The mass flux out of one is identically the mass flux into the other.

$$\oiint_S (\rho V \cdot ds)\, u = -\int_i^a \rho_1 u_1^2\, dy + \int_h^b \rho_2 u_2^2\, dy$$

Because V and ds being opposite direction along ai

Continuity equation

$$\oiint_S (\rho V) ds = -\int_i^a \rho_1 u_1\, dy + \int_h^b \rho_2 u_2\, dy = 0$$

$$\Rightarrow \int_i^a \rho_1 u_1\, dy = \int_h^b \rho_2 u_2\, dy \qquad \text{multiplying } u_1$$

$$\int_i^a \rho_1 u_1^2\, dy = \int_h^b \rho u_1 u_2\, dy$$

$$\oiint_S (\rho V \cdot ds)\, u = -\int_h^b \rho_2 u_2 u_1 \, dy + \int_h^b \rho_2 u_2^2 \, dy$$

$$\oiint_S (\rho V \cdot ds)\, u = -\int_h^b \rho_2 u_2 (u_1 - u_2)\, dy$$

$$\boxed{D' = \int_h^b \rho_2 u_2 (u_1 - u_2)\, dy}$$

$\rho_2 u_2$: mass flux

$u_1 - u_2$: Velocity decrement at a given y location.

Decrement in momentum by $(u_1 - u_2)$ and $\rho_2 u_2$ mass flux that exits across the wake.

For incompressible flow.

$$\boxed{D' = \rho \int_h^b u_2 (u_1 - u_2)\, dy}$$

Ex.

Consider an incompressible flow laminar boundary layer growing along the surface flat plate, with chord length c. The thickness of the B.L. at trailing edge.

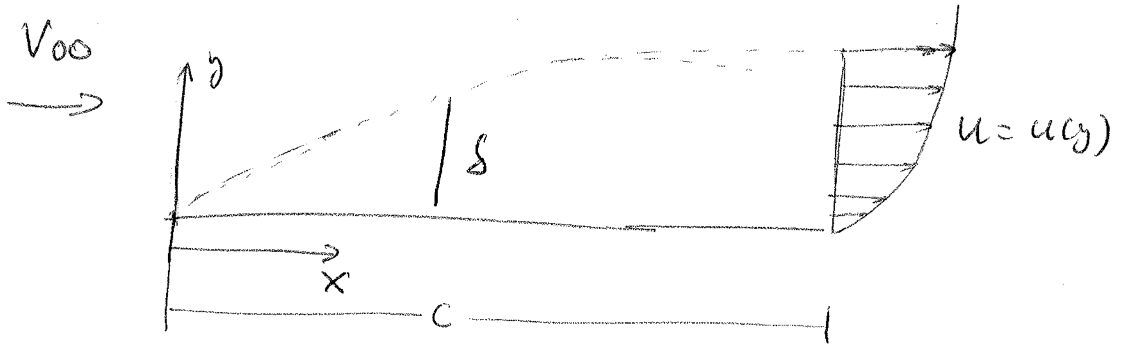
$$\delta = \frac{5c}{(Re_c)^{1/2}}$$

the skin friction coefficient for the plate is

$$C_f = \frac{D'}{q_\infty c(1)} = \frac{1.328}{\sqrt{Re_c}}$$

assuming velocity profile through the boundary layer is given by power-law

$$u = V_\infty \left(\frac{y}{\delta}\right)^n$$



Calculate the value of n, consistent with the all information given above

$$C_f = \frac{D'}{q_\infty c} = \frac{\rho_\infty}{\frac{1}{2}\rho_\infty V_\infty^2 c} \int_0^\delta u_2 (u_1 - u_2)\, dy$$

$$C_f = \frac{2\rho_\infty}{\rho_\infty} \int_0^{\delta/c} \frac{u_2}{V_\infty}\left(\frac{u_1}{V_\infty} - \frac{u_2}{V_\infty}\right) d\left(\frac{y}{c}\right)$$

$$C_f = 2 \int_0^{\delta/c} \frac{u_2}{V_\infty}\left(1 - \frac{u_2}{V_\infty}\right) d\left(\frac{y}{c}\right)$$

$$\frac{1.328}{\sqrt{Re_c}} = 2 \int_0^{\delta/c} \cancel{\frac{V_\infty}{V_\infty}} \left(\frac{y/c}{\delta/c}\right)^n \left[1 - \cancel{\frac{V_\infty}{V_\infty}}\left(\frac{y/c}{\delta/c}\right)^n\right] d(y/c)$$

$$\frac{1.328}{\sqrt{Re_c}} = 2 \int_0^{\delta/c} \left[\left(\frac{y/c}{\delta/c}\right)^n - \left(\frac{y/c}{\delta/c}\right)^{2n}\right] d(y/c)$$

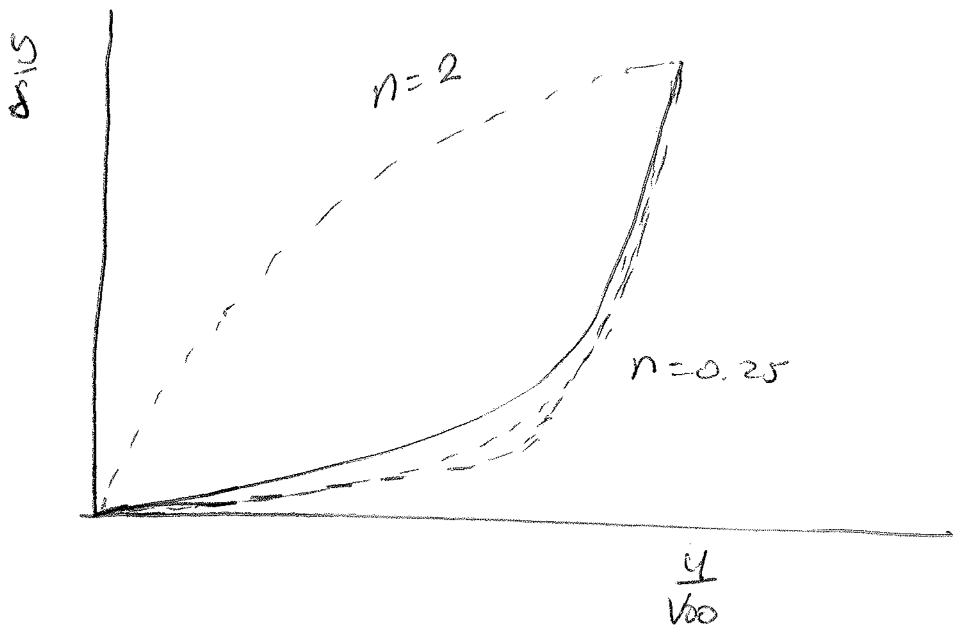
$$\frac{1.328}{\sqrt{Re_c}} = \frac{2}{n+1}\left(\frac{\delta}{c}\right) - \frac{2}{2n+1}\left(\frac{\delta}{c}\right)$$

$$\frac{1.328}{\sqrt{Re_c}} = \frac{10}{n+1}\,\frac{1}{\sqrt{Re_c}} - \frac{10}{2n+1}\left(\frac{1}{\sqrt{Re_c}}\right)$$

$$\frac{1}{n+1} - \frac{1}{2n+1} = \frac{1.328}{10}$$

$$0.2656n^2 - 0.6016n + 0.1328 = 0$$

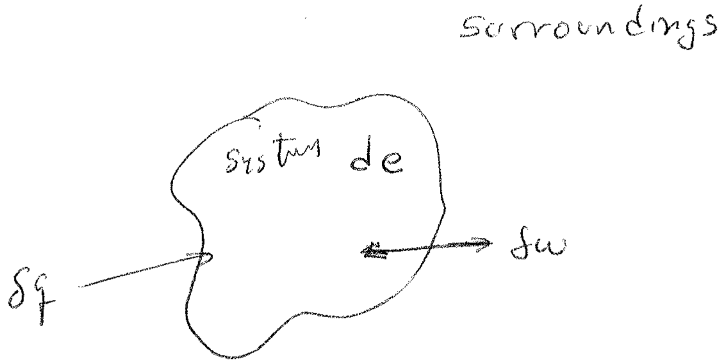
$$n = 2 \quad \text{or} \quad n = 0.25$$



Energy Equation

Physical Principle: Energy can be neither nor destroyed: It can only change the form.

Considos a fixed amount of matter contained within a closed boundary. This matter defines the system. This system contains a certain amount of energy. The system contain a unit mass. (e) internal energy per unit mass.



$$\delta q + \delta w = de$$ 1st Law of thermodynamics.

We apply the 1st Law to the fluid flowing through the fixed control volume.

$$B_1 + B_2 = B_3$$

B_1 = rate of heat added To fluid inside C.V. from surrounding.

B_2 = rate of work done on the system (fluid)

B_3 = rate of change of energy of fluid.

B_1 = This can be visualized as volumetric heating of the fluid inside C.V. due to the absorption of radiation origining outside the system

- mass contained within a elemental volume is $\rho\, dV$.
- rate of heat addition to this mass is $\dot{q}(\rho\, dV)$

Rate of volumetric heating $= \oiiint \dot{q}\, \rho\, dV$

Note! If flow is viscous. Heat can be transferred into C.V. by means of thermal conduction and mass diffussion across the C. Surface.

$$B_1 = \oiiint \dot{q}\, \rho\, dV + \dot{Q}_{viscous}.$$

Rate of doing work on moving body $= F.\vec{V}$; This result leads to an expression for B_2. Consider a ds area of the control surface. Pressure force on this elemental area is $P\,dS$. The work due to this force is : $= (P\,ds)\vec{V}$

Rate of work done on fluid inside Voulume due to pressure force on S $= -\oiint (P\, d\vec{s})\, \vec{V}$

Consider the elemental volume dV inside control volume. f is the body force per unit mass. Rate of work done on the elemental volume due to the body force $(\rho f\, dV)\vec{V}$.

Rate of work done on fluid inside V due to body forces $= \iiint_V (\rho f\, dV).\vec{V}$

Note: If the flow is viscous, the shear stress on the control ~~surface~~ (surface) will also perform work on the fluid as it passes across the surface.

$$B_2 = -\oiint_S (P\, ds)\vec{V} + \iiint_V (\rho f \vec{V})\, dV + \dot{W}_{viscous}.$$

To visualize the energy inside the control volume, the internal energy (e) is due to the random motion of atoms and molecules inside de system. Applying the 1st Law for open system, ~~it~~ it needs to consider the kinetic energy. The total energy $e + \frac{V^2}{2}$

Net rate of flow of total energy across C. Surface $= \oiint_S (\rho \vec{V} ds).\left(e + \frac{V^2}{2}\right)$

If flow is unsteady, there is a time rate of change of total energy inside the control volume, due to the transient fluctuations of the flow-field variables.

Total energy in elemental volume dV is $\rho\left(e+\frac{V^2}{2}\right)dV$

The complete energy inside C.V. at any instant in time is

$$\iiint_V \rho\left(e+\frac{V^2}{2}\right)dV$$

Time rate of change of total energy inside V due to transient variations of flow-field variables $= \frac{\partial}{\partial t}\iiint_V \rho\left(e+\frac{V^2}{2}\right)dV$

$$B_3 = \frac{\partial}{\partial t}\iiint_V \rho\left(e+\frac{V^2}{2}\right)dV + \iint_S (\rho\vec{V}\cdot ds)\left(e+\frac{V^2}{2}\right)$$

$$\iiint_V \dot{q}\rho\, dv + \dot{Q}_{viscous} - \iint_S p\vec{V}\cdot ds + \iiint_V \rho(f)\vec{V}\,dV + \dot{W}_{viscous} =$$

$$\frac{\partial}{\partial t}\iiint_V \rho\left(e+\frac{V^2}{2}\right)dV + \iint_S (\rho\vec{V}\cdot ds)\left(e+\frac{V^2}{2}\right)$$

$$\iiint \left(\frac{\partial}{\partial t}\left[\rho\left(e+\frac{V^2}{2}\right)\right] + \nabla\left[\rho\left(e+\frac{V^2}{2}\right)\vec{V}\right] + \dot{q}\rho\,dv - \nabla\cdot(P\vec{V}) + \rho(f\cdot\vec{V}) + \dot{Q}_{viscous} + \dot{W}_{viscous} \right) dV$$

For ~~steady~~ steady, Inviscid flow and adiabatic we obtain

$$\oiint_S \rho\left(e+\frac{V^2}{2}\right)\vec{V}\cdot ds = -\oiint P\vec{V}\cdot ds + \rho(fV)$$

for $f=0$; we obtain $\oiint_S \rho\left(e+\frac{V^2}{2}\right)\vec{V}\,ds = -\oiint(P\vec{V}\cdot ds)$

we obtain $\nabla\cdot\left[\rho\left(e+\frac{V^2}{2}\right)\cdot\vec{V}\right] = -\nabla\cdot(P\vec{V})$

Energy equation. For incompressible flow, thermal conductivity constant

$$\rho C_p \frac{DT}{Dt} = k\nabla^2 T + \Phi$$

where $\Phi = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_i}{\partial x_j} + \lambda\left(\frac{\partial u_i}{\partial x_i}\right)^2$ $\quad \lambda = -\frac{2}{3}\mu$

$$\Phi = \mu\left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\cdot\frac{\partial u}{\partial y} + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial x} + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\frac{\partial w}{\partial x} + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\frac{\partial u}{\partial z} + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\frac{\partial w}{\partial y} + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\frac{\partial v}{\partial z} + 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2\right]$$
$$+ \lambda\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right]^2$$

$$\Phi = \mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2\right] + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2$$