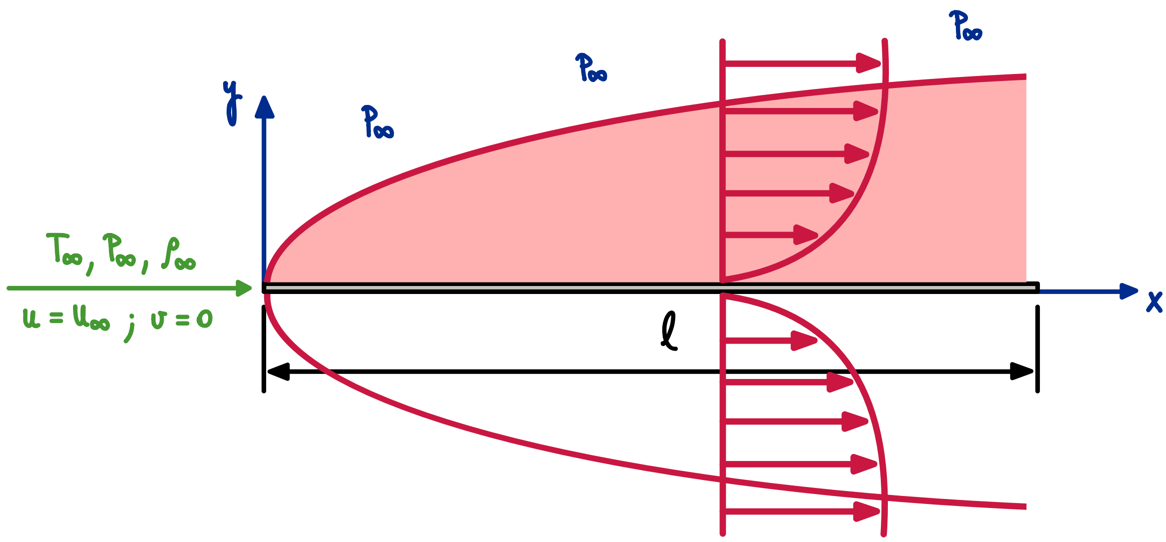
Capa Límite de Blasius

La solución de Blasius corresponde a la del estudio de la capa límite generada sobre una placa a ángulo de ataque nulo sometida a un flujo de velocidad constante U_∞ y por tanto sin gradiente exterior de presiones:



Euler : $\begin{cases} \vec{V}_E = U_\infty \vec{i} \\ \dfrac{P_E - P_\infty}{\rho} = 0 \ \forall x \end{cases} \xrightarrow{\text{Sobre la placa}} \boxed{\begin{array}{c} U_e = U_\infty \\ \dfrac{dU_e}{dx} = 0 \end{array}}$ SOLUCIÓN EXTERIOR

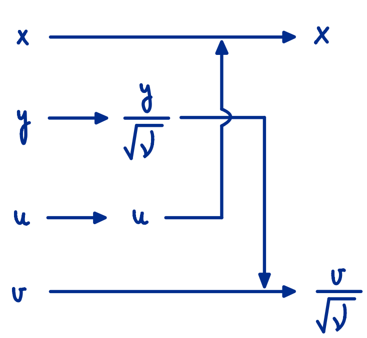
Al no haber gradiente de presiones, el problema de capa límite es:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad\qquad y = 0: \ u = 0 \ ; \ v = 0$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \qquad\qquad y \to \infty: \ u \to U_\infty$$

$$x = 0: \ u = 0 \ \forall y \ \text{(Todavía no hay C.L.)}$$

Nos libramos del pámetro ν:



El problema pasa a ser:

$$\frac{\partial u}{\partial x} + \frac{\partial\left(\frac{v}{\sqrt{\nu}}\right)}{\partial\left(\frac{y}{\sqrt{\nu}}\right)} = 0$$

$$u\frac{\partial u}{\partial x} + \frac{v}{\sqrt{\nu}}\frac{\partial\left(\frac{v}{\sqrt{\nu}}\right)}{\partial\left(\frac{y}{\sqrt{\nu}}\right)} = \nu\frac{\partial^2 u}{\partial\left(\frac{y}{\sqrt{\nu}}\right)^2}$$

$$\frac{y}{\sqrt{\nu}} = 0: \quad u = 0 \; ; \; \frac{v}{\sqrt{\nu}} = 0$$

$$\frac{y}{\sqrt{\nu}} \to \infty: \quad u \to U_\infty$$

$$x = 0: \quad u = 0 \quad \forall \; \frac{y}{\sqrt{\nu}}$$

$$\longrightarrow \boxed{\begin{gathered} u = u\left(x, \frac{y}{\sqrt{\nu}}, U_\infty\right) \\ \frac{v}{\sqrt{\nu}} = \frac{v}{\sqrt{\nu}}\left(x, \frac{y}{\sqrt{\nu}}, U_\infty\right) \end{gathered}}$$

Como simplificación adicional introducimos la función de corriente ψ, de forma que pasamos de trabajar con 2 VD (u, v) a trabajar con 1 VD (ψ). Al introducir esta función la ecuación de continuidad se satisface automáticamente.

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial\left(\frac{\psi}{\sqrt{\nu}}\right)}{\partial\left(\frac{y}{\sqrt{\nu}}\right)}$$

$$v = -\frac{\partial \psi}{\partial x} \longrightarrow \frac{v}{\sqrt{\nu}} = -\frac{\partial\left(\frac{\psi}{\sqrt{\nu}}\right)}{\partial x}$$

$$\longrightarrow \frac{\psi}{\sqrt{\nu}} = \frac{\psi}{\sqrt{\nu}}\left(x, \frac{y}{\sqrt{\nu}}, U_\infty\right)$$

Ecuaciones de dimensiones:

$$u = \frac{\partial \psi}{\partial y} \longrightarrow [\psi] = [u][x] = L^2T^{-1}$$

$$[Re] = \frac{[u][x]}{[\nu]} = 1 \longrightarrow [\nu] = [u][x] = L^2T^{-1}$$

$$\left[\frac{\psi}{\sqrt{\nu}}\right] = L^2T^{-1}\left(L^2T^{-1}\right)^{-1/2} = LT^{-1/2}$$

$$[x] = L$$

$$\left[\frac{y}{\sqrt{\nu}}\right] = L\left(L^2T^{-1}\right)^{-1/2} = T^{1/2}$$

$$[U_\infty] = LT^{-1}$$

$\longrightarrow$ 2 magnitudes dimensionalmente independientes

$$\boxed{U_\infty \; ; \; x}$$

Adimensionalizamos las magnitudes:

$$\frac{\frac{\psi}{\sqrt{\nu}}}{x^{\alpha} U_\infty^{\beta}} \rightarrow \frac{L T^{-1/2}}{L^{\alpha+\beta} T^{-\beta}} \rightarrow \begin{cases} -\frac{1}{2} = -\beta \\ \alpha + \beta = 1 \end{cases} \rightarrow \begin{cases} \alpha = \frac{1}{2} \\ \beta = \frac{1}{2} \end{cases} \rightarrow \frac{\psi}{\sqrt{U_\infty \nu x}}$$

$$\frac{\frac{y}{\sqrt{\nu}}}{x^{\alpha} U_\infty^{\beta}} \rightarrow \frac{T^{1/2}}{L^{\alpha+\beta} T^{-\beta}} \rightarrow \begin{cases} \frac{1}{2} = -\beta \\ \alpha + \beta = 0 \end{cases} \rightarrow \begin{cases} \alpha = \frac{1}{2} \\ \beta = -\frac{1}{2} \end{cases} \rightarrow y\sqrt{\frac{U_\infty}{\nu x}}$$

Por tanto:

$$\frac{\psi}{\sqrt{U_\infty \nu x}} = f\left(y\sqrt{\frac{U_\infty}{\nu x}} \right)$$

$$\frac{\psi}{\sqrt{2 U_\infty \nu x}} = f(\eta) \quad , \text{ donde } \boxed{\eta = y\sqrt{\frac{U_\infty}{2\nu x}}} \rightarrow \psi = \sqrt{2 U_\infty \nu x}\, f(\eta)$$

Simplificará la EDO que vamos a obtener

Derivadas de la variable de semejanza:

$$\frac{\partial \eta}{\partial x} = y\sqrt{\frac{U_\infty}{2\nu}} \cdot \left(-\frac{1}{2}\right) x^{-3/2} \rightarrow \boxed{\frac{\partial \eta}{\partial x} = -\frac{1}{2}\frac{\eta}{x}} \; ; \; \boxed{\frac{\partial \eta}{\partial y} = \sqrt{\frac{U_\infty}{2\nu x}}}$$

Como la ecuación de continuidad se satisface automáticamente por trabajar con la función de corriente, nos centramos en la $ECdM_x$:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

$\boxed{u}$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left[\sqrt{2U_\infty \nu x}\, f\right] = \sqrt{2U_\infty \nu x}\,\frac{\partial f}{\partial y} = \sqrt{2U_\infty \nu x}\,\underbrace{\frac{\partial f}{\partial \eta}}_{f'}\frac{\partial \eta}{\partial y} = \sqrt{2U_\infty \nu x}\, f' \sqrt{\frac{U_\infty}{2\nu x}} \rightarrow$$

$$\rightarrow \boxed{u = U_\infty f'}$$

$\boxed{v}$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left[\sqrt{2U_\infty \nu x}\; f\right] = -\sqrt{2U_\infty \nu}\left(\frac{1}{2}x^{-1/2} f + \sqrt{x}\, f' \frac{\partial \eta}{\partial x}\right) = -\sqrt{2U_\infty \nu}\left(\frac{1}{2}x^{-1/2} f - \right.$$

$$\left. - \sqrt{x}\, f' \frac{1}{2}\frac{\eta}{x}\right) = -\sqrt{2U_\infty \nu}\left(\frac{1}{2}x^{-1/2} f + \frac{1}{2}x^{-1/2} f' \eta\right) = \sqrt{2U_\infty \nu}\; \frac{x^{-1/2}}{2}\left(\eta f' - f\right) \longrightarrow$$

$$\longrightarrow \boxed{v = \sqrt{\frac{U_\infty \nu}{2x}}\left(\eta f' - f\right)}$$

$\boxed{\dfrac{\partial u}{\partial x}}$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(U_\infty f'\right) = U_\infty f'' \frac{\partial \eta}{\partial x} = U_\infty f''\left(-\frac{1}{2}\frac{\eta}{x}\right) \longrightarrow \boxed{\frac{\partial u}{\partial x} = -\frac{U_\infty}{2}\frac{\eta}{x} f''}$$

$\boxed{\dfrac{\partial u}{\partial y}}$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(U_\infty f'\right) = U_\infty f'' \frac{\partial \eta}{\partial y} = U_\infty f'' \sqrt{\frac{U_\infty}{2\nu x}} \longrightarrow \boxed{\frac{\partial u}{\partial y} = U_\infty \sqrt{\frac{U_\infty}{2\nu x}}\, f''}$$

$\boxed{\dfrac{\partial^2 u}{\partial y^2}}$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(U_\infty \sqrt{\frac{U_\infty}{2\nu x}}\, f''\right) = U_\infty \sqrt{\frac{U_\infty}{2\nu x}}\, f''' \frac{\partial \eta}{\partial y} = U_\infty \sqrt{\frac{U_\infty}{2\nu x}}\, f''' \sqrt{\frac{U_\infty}{2\nu x}} \longrightarrow$$

$$\longrightarrow \boxed{\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty^2}{2\nu x} f'''}$$

Sustituyendo en la $ECDM_x$:

$$U_\infty f'\left(-\frac{U_\infty}{2}\frac{\eta}{x} f''\right) + \sqrt{2U_\infty \nu}\; \frac{x^{-1/2}}{2}\left(\eta f' - f\right) U_\infty \sqrt{\frac{U_\infty}{2\nu x}}\, f'' = \nu \frac{U_\infty^2}{2\nu x} f'''$$

$$\cancel{-\frac{U_\infty^2}{2x}\eta f' f''} + \cancel{\frac{U_\infty^2}{2x}\eta f' f''} - \frac{U_\infty^2}{2x} f f'' = \frac{U_\infty^2}{2x} f'''$$

$$\boxed{f''' + f f'' = 0}$$

Respecto a las condiciones de contorno :

$$y=0:\ u=0\ ;\ v=0 \qquad\qquad \eta=0:\ f=0\ ;\ f'=0$$
$$y\to\infty:\ u\to U_\infty \quad\Longrightarrow\quad \left.\begin{array}{l}\eta\to\infty:\ f'\to 1\\ \eta\to\infty:\ f'\to 1\end{array}\right\}\ \text{COLAPSAN}$$
$$x=0:\ u=0\ \forall y$$

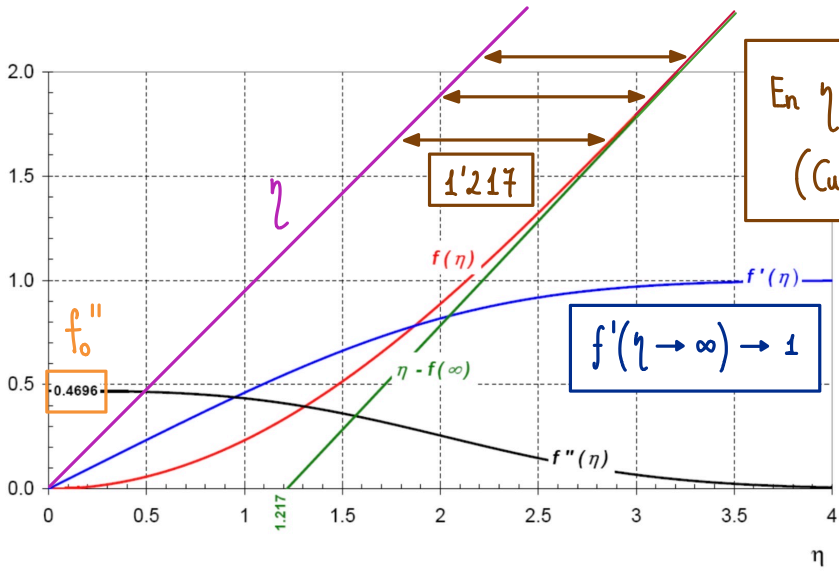
El problema queda :

$$\boxed{\begin{array}{l} f'''+ff''=0\\ \eta=0:\ f=0\ ;\ f'=0\\ \eta\to\infty:\ f'\to 1\end{array}} \xrightarrow{\text{VALOR DE CONTORNO} \,\to\, \text{VALOR INICIAL}} \boxed{\begin{array}{l} f'''+ff''=0\\ \eta=0:\left\{\begin{array}{l} f=0\ ;\ f'=0\\ f''=f''_0\ \Big/\ \lim\limits_{\eta\to\infty} f'=1\end{array}\right.\end{array}}$$

(BLASIUS, 1908)

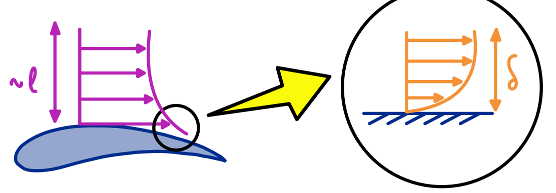
Representación gráfica de la solución :

$$f\big|_{\eta\to\infty}=\eta+C\ \ (\text{RECTA}) \longrightarrow (f-\eta)_{\eta\to\infty}=cte$$



$$\text{En } \eta\to\infty:\ f-\eta=cte\longrightarrow 1'217$$
(Cuando f se hace lineal con η)

NOTA



En la escala de la C.L. : $\dfrac{dU_e}{dx}\approx 0$

$$\frac{v}{U_\infty}\Big|_{\eta\to\infty}=\sqrt{\frac{\nu}{2U_\infty x}}\,(\eta f'-f)_{\eta\to\infty} \overset{\eta\to\infty:\ f'\to 1}{=} \sqrt{\frac{\nu}{2U_\infty x}}\,\overbrace{(\eta-f)_{\eta\to\infty}}^{1'217}\longrightarrow \boxed{\frac{v}{U_\infty}\Big|_{\eta\to\infty}=0'86\ Re_x^{-1/2}}$$

Ángulo de ataque de la línea de corriente :

$$\alpha\big|_{\eta\to\infty}=\text{arctg}\left(\frac{v}{u}\right)\Big|_{\eta\to\infty}\approx\frac{v}{u}\Big|_{\eta\to\infty}\approx\frac{v\big|_{\eta\to\infty}}{U_\infty}\longrightarrow \boxed{\alpha\big|_{\eta\to\infty}\approx 0'86\ Re_x^{-1/2}}$$

Espesor de desplazamiento :

$$dy\Big|_{x=cte} = \sqrt{\frac{2\nu x}{U_\infty}}\, d\eta$$

$$\delta_1 = \int_0^\infty \Big(1 - \underbrace{\frac{u}{U_e}}_{f'}\Big)\, dy = \sqrt{\frac{2\nu x}{U_\infty}} \int_0^\infty \big(1 - f'\big)\, d\eta = \sqrt{\frac{2\nu x}{U_\infty}}\ \underbrace{\big(\eta - f\big)_{\eta\to\infty}}_{1'217} = 1'72\sqrt{\frac{\nu x}{U_\infty}} \longrightarrow$$

$$\longrightarrow \frac{\delta_1}{x} = 1'72\sqrt{\frac{\nu}{U_\infty x}} \longrightarrow \boxed{\frac{\delta_1}{x} = 1'72\ Re_x^{-1/2}}$$

Espesor de cantidad de movimiento :

$$f(0) = 0$$

$$\delta_2 = \int_0^\infty \frac{u}{U_e}\Big(1 - \frac{u}{U_e}\Big)\, dy = \sqrt{\frac{2\nu x}{U_\infty}} \int_0^\infty f'\big(1 - f'\big)\, d\eta = \sqrt{\frac{2\nu x}{U_\infty}} \left[f\Big|_0^\infty - \int_0^\infty (f')^2\, d\eta \right] =$$

$$= \sqrt{\frac{2\nu x}{U_\infty}} \left[f\Big|_{\eta\to\infty} - \underbrace{\int_0^\infty (f')^2\, d\eta}_{?} \right] \qquad \left\{ \begin{array}{l} u = f' \longrightarrow du = f''\, d\eta \\ dv = f'\, d\eta \longrightarrow v = f \end{array} \right\}$$

$$f''' + ff'' = 0 \longrightarrow ff'' = -f''' \qquad\qquad f''\Big|_{\eta\to\infty} = 0$$

$$\int_0^\infty (f')^2\, d\eta = ff'\Big|_0^\infty - \int_0^\infty ff''\, d\eta = f\Big|_{\eta\to\infty} + \int_0^\infty f'''\, d\eta = f\Big|_{\eta\to\infty} - f''_0$$

Por tanto :

$$\delta_2 = \sqrt{\frac{2\nu x}{U_\infty}} \left(\cancel{f\Big|_{\eta\to\infty}} - \cancel{f\Big|_{\eta\to\infty}} + f''_0 \right) \longrightarrow \delta_2 = \sqrt{\frac{2\nu x}{U_\infty}}\, f''_0$$

$$\boxed{\frac{\delta_2}{x} = 0'66\ Re_x^{-1/2}}$$

Obtenidos δ_1 y δ_2 podemos obtener el factor de forma :

$$H_{12} = \frac{\delta_1}{\delta_2} = \frac{\frac{\delta_1}{x}}{\frac{\delta_2}{x}} = \frac{1'72\ Re_x^{-1/2}}{0'66\ Re_x^{-1/2}} = 2'59$$

$$\boxed{H_{12} = 2'59}$$

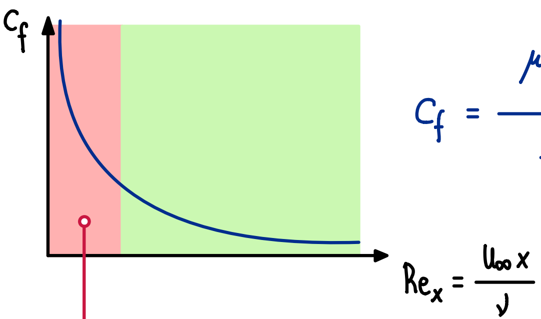
Respecto al coeficiente de fricción :

$$\tau_p = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu U_\infty \left.\frac{\partial f'}{\partial y}\right|_{y=0} = \mu U_\infty \underbrace{\left.\frac{df'}{d\eta}\right|_{\eta=0}}_{f_0''} \underbrace{\left.\frac{\partial \eta}{\partial y}\right|_{\eta=0}}_{\sqrt{\frac{U_\infty}{2\nu x}}} \longrightarrow \tau_p = \mu U_\infty \sqrt{\frac{U_\infty}{2\nu x}}\, f_0''$$

$$C_f(x) = \frac{\tau_p(x)}{\frac{1}{2}\rho U_e^2(x)} = \frac{\tau_p(x)}{\frac{1}{2}\rho U_\infty^2} = \frac{\mu U_\infty \sqrt{\frac{U_\infty}{2\nu x}}\, f_0''}{\frac{1}{2}\rho U_\infty^2} = \sqrt{\frac{2\nu}{U_\infty x}}\, f_0'' = \sqrt{2}\, \underbrace{f_0''}_{0'4696 \text{ (ver gráfica)}} Re_x^{-1/2}$$

$$\boxed{C_f(x) = 0'664\, Re_x^{-1/2}}$$

Hay un problema : singularidad en el origen



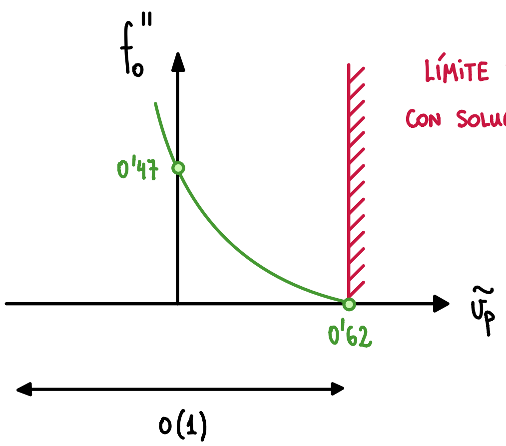
$$C_f = \frac{\mu \left.\frac{\partial u}{\partial y}\right|_{y=0}}{\frac{1}{2}\rho U_\infty^2} \sim \frac{\mu \frac{U_\infty}{\delta}}{\rho U_\infty^2} \sim \frac{\nu}{U_\infty} \frac{Re_x^{-1/2}}{x} \sim Re_x^{-1/2}$$

$$\frac{\delta_2}{x} \sim Re_x^{-1/2}$$

$\longrightarrow$ C_f y δ_2 ESCALAN IGUAL

$Re \sim 100$ SOLUCIÓN DE C.L. NO VÁLIDA PORQUE EL FLUJO NO ES ESBELTO $\longrightarrow \frac{\delta}{\ell} \ll 1 \longrightarrow \frac{\delta}{x} \ll 1 \;;\; \frac{\delta}{x} \sim Re_x^{-1/2} \longrightarrow Re_x^{1/2} \gg 1$

VALE SI EL FLUJO ES ESBELTO : $Re_x \gtrsim 100$



LÍMITE SUCCIÓN/SOPLADO CON SOLUCIÓN DE SEMEJANZA

$$\tau_p = \mu U_\infty \sqrt{\frac{U_\infty}{2\nu x}}\, f_0'' \longrightarrow \boxed{f_0'' \rightleftharpoons \tau_p}$$

Suponiendo la placa plana POROSA, la solución de C.L. sigue siendo autosemejante si la velocidad de succión/soplado es de la forma:

$$v\Big|_{y=0} = v_p = \sqrt{\frac{\nu}{U_\infty x}}\, U_\infty \tilde{v}_p \;, \text{ con } \tilde{v}_p = O(1) \text{ conocido}$$

$v_p = \sqrt{\frac{\nu}{U_\infty x}}\, U_\infty \tilde{v}_p = \tilde{v}_p U_\infty Re_x^{-1/2}$ Me da una condición para $f(0)$ (antes era $f(0) = 0$ por ser $v_p = 0$)

$v = \sqrt{\frac{U_\infty \nu}{2x}}\left(\eta f' - f\right) \longrightarrow$ En $\eta = 0$: $v\Big|_{\eta=0} = -\sqrt{\frac{U_\infty \nu}{2x}}\, f(0) = -\frac{U_\infty}{\sqrt{2}} Re_x^{-1/2} f(0)$

Igualando ambas expresiones:

$$\tilde{v}_p U_\infty Re_x^{-1/2} = -\frac{U_\infty}{\sqrt{2}} Re_x^{-1/2} f(0)$$

$$\boxed{f(0) = -\sqrt{2}\,\tilde{v}_p}$$

Da lo mismo que si imponemos la ley de v_p y $f' = 1$ en la expresión de v.

Y el problema pasa a ser:

$$\boxed{\begin{aligned} & f''' + f f'' = 0 \\ & \eta = 0: \; f = -\sqrt{2}\,\tilde{v}_p \;;\; f' = 0 \\ & f'' = f_0''(\tilde{v}_p) \Big/ \lim_{\eta \to \infty} f' = 1 \end{aligned}}$$

SOLUCIONES DE SEMEJANZA DE C.L. SOBRE PLACA PLANA CON SUCCIÓN ($\tilde{v}_p < 0$) O SOPLADO ($\tilde{v}_p > 0$)