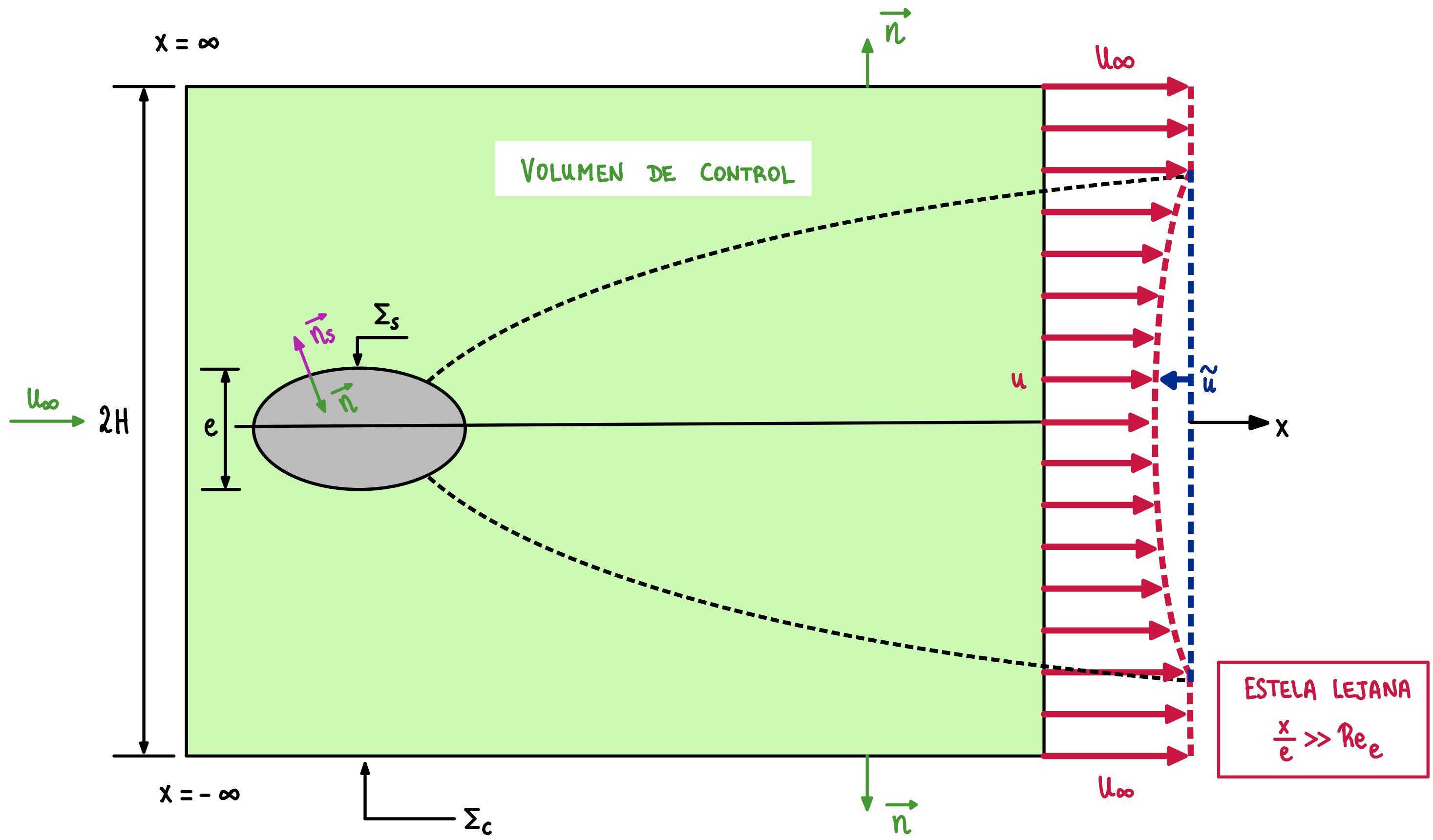
Estela Plana Laminar



$$\boxed{u(x,y) = U_\infty - \tilde{u}(x,y)}$$

Estela lejana: $Re_e = \dfrac{U_\infty e}{\nu}$

$$\boxed{\frac{U_\infty - u}{U_\infty} = \frac{\tilde{u}}{U_\infty} \ll 1 \quad \left(\frac{x}{e} \gg Re_e\right)}$$

$$\frac{\partial \tilde{u}}{\partial \blacksquare} = -\frac{\partial u}{\partial \blacksquare} \longrightarrow \text{Ecuación de continuidad:} \quad \underbrace{\frac{\partial \tilde{u}}{\partial x}}_{\sim \frac{\tilde{u}_c}{x}} - \underbrace{\frac{\partial v}{\partial y}}_{\sim \frac{v_c}{\delta}} = 0 \longrightarrow \boxed{\frac{v_c}{\tilde{u}_c} \sim \frac{\delta}{x} \ll 1}$$

$$\boxed{v_c \ll \tilde{u}_c \ll U_\infty}$$

Ecuación de continuidad aplicada al volumen de control:

$$\int_{\Sigma_c} \vec{v}\,\vec{n}\, dA = 0 \longrightarrow -U_\infty 2H + 2\int_{-\infty}^{x} v\, dx + 2\int_0^H u\, dy = 0$$

ECdM$_x$ aplicada al volumen de control:

$$\int_{\Sigma_c} \rho \vec{v}(\vec{v}\,\vec{n})\, dA\, \vec{i} = \underbrace{-\int_{\Sigma_s} (p - p_\infty)\,\vec{n}\, dA\, \vec{i} + \int_{\Sigma_c} \overline{\overline{\tau}}_\mu \cdot \vec{n}\, dA\, \vec{i}}_{-D} \longrightarrow -U_\infty^2 2H + 2U_\infty \int_{-\infty}^{x} v\, dx + 2\int_0^H u^2 dy = -\frac{D}{\rho}$$

D: resistencia aerodinámica del cuerpo

Combinando ambas ecuaciones:

$$\left.\begin{aligned} &-U_\infty 2H + 2\int_{-\infty}^{x} v\,dx + 2\int_0^H u^2\,dy = 0 \\ &-U_\infty^2 2H + 2U_\infty\int_{-\infty}^{x} v\,dx + 2\int_0^H u\,dy = -\frac{D}{\rho} \end{aligned}\right\} \longrightarrow \int_0^H \left(u - \frac{u^2}{U_\infty}\right) dy = \frac{D}{2\rho U_\infty} \longrightarrow$$

$$\longrightarrow \int_0^H u\left(1 - \frac{u}{U_\infty}\right) dy = \frac{D}{2\rho U_\infty} \longrightarrow \int_0^H u\,(U_\infty - u)\,dy = \frac{D}{2\rho} \longrightarrow \frac{D}{\rho} = 2\int_0^H u\,(U_\infty - u)\,dy$$

$$\boxed{\frac{D}{\rho} = 2\int_0^H u\,(U_\infty - u)\,dy = 2\int_0^H u\,\tilde{u}\,dy}$$

En el exterior de la estela $(U_\infty - u) \rightarrow 0$, de modo que no contribuye a la integral anterior $(H \rightarrow \infty)$:

$$\forall\, x \in \text{estela lejana}: \int_0^\infty u\,\tilde{u}\,dy = \int_0^\infty \left(U_\infty - \cancel{\tilde{u}}\right)\tilde{u}\,dy \simeq \int_0^\infty U_\infty\,\tilde{u}\,dy = \frac{D}{2\rho} \longrightarrow \boxed{\frac{D}{\rho U_\infty} = 2\int_0^\infty \tilde{u}\,dy}$$

$\ll$

Como consecuencia de esta disparidad de escalas podemos linealizar la $ECdM_x$:

$\ll$

$$\underbrace{u\,\frac{\partial \tilde{u}}{\partial x}}_{\sim U_\infty \frac{\tilde{u}_c}{x}} + \underbrace{\cancel{v\,\frac{\partial \tilde{u}}{\partial y}}}_{\sim v_c \frac{\tilde{u}_c}{\delta}} = \underbrace{\nu\,\frac{\partial^2 \tilde{u}}{\partial y^2}}_{\sim \nu \frac{\tilde{u}_c}{\delta^2}}$$

$$\sim \frac{\tilde{u}_c^2}{x}$$

$$\sim \frac{\nu x}{U_\infty}\left(\frac{x}{\delta}\right)^2 \sim Re_x^{-1}\left(\frac{\delta}{x}\right)^{-2}$$

$$\sim \frac{\tilde{u}_c^2}{x}\,\frac{x}{U_\infty\,\tilde{u}_c} \sim \frac{\tilde{u}_c}{U_\infty} \ll 1$$

$$1$$

$$\boxed{\frac{\delta}{x} \sim Re_x^{-1/2} \ll 1}$$

$\ll$

La ecuación queda: $u\,\dfrac{\partial \tilde{u}}{\partial x} = \nu\,\dfrac{\partial^2 \tilde{u}}{\partial y^2} \longrightarrow U_\infty\left(1 - \cancel{\dfrac{\tilde{u}}{U_\infty}}\right)\dfrac{\partial \tilde{u}}{\partial x} = \nu\,\dfrac{\partial^2 \tilde{u}}{\partial y^2} \longrightarrow \boxed{U_\infty\,\dfrac{\partial \tilde{u}}{\partial x} \approx \nu\,\dfrac{\partial^2 \tilde{u}}{\partial y^2}}$

EDP LINEAL

En cuanto a las condiciones de contorno:

- Simetría: $\begin{cases} \tilde{u}(y) = \tilde{u}(-y) \\ \left.\dfrac{\partial \tilde{u}}{\partial y}\right|_{y=0} = \lim\limits_{\Delta y \to 0} \dfrac{\tilde{u}(\Delta y/2) - \tilde{u}(-\Delta y/2)}{\Delta y} = 0 \longrightarrow y=0: \dfrac{\partial \tilde{u}}{\partial y} = 0 \end{cases}$

- En $y \to \infty$ la estela no ejerce ninguna influencia: $y \to \infty : u \to U_\infty \longrightarrow y \to \infty : \tilde{u} \to 0$

- Además sabemos que $\forall x \in$ estela lejana: $\dfrac{D}{2\rho U_\infty} = \displaystyle\int_0^\infty \tilde{u}\, dy$

Adicionalmente, de la ecuación de continuidad: $\left.\begin{array}{l} \dfrac{\partial v}{\partial y} = \dfrac{\partial \tilde{u}}{\partial x} \\ v(y) = v(-y) \longrightarrow y=0 : v = 0 \end{array}\right\} \longrightarrow \boxed{v = \dfrac{d}{dx}\displaystyle\int_0^y \tilde{u}\, dy}$

UNA VEZ CONOZCAMOS $\tilde{u}$ SACAMOS v DE AQUÍ

El problema queda:

$$\boxed{\begin{array}{c} U_\infty \dfrac{\partial \tilde{u}}{\partial x} \approx \nu \dfrac{\partial^2 \tilde{u}}{\partial y^2} \\ y=0 : \dfrac{\partial \tilde{u}}{\partial y} = 0 \\ y \to \infty : \tilde{u} \to 0 \\ \forall x : \dfrac{D}{2\rho U_\infty} = \displaystyle\int_0^\infty \tilde{u}\, dy \end{array}} \xrightarrow[x,\ \sqrt{\frac{U_\infty}{\nu}}y,\ \tilde{u}]{\text{Eliminamos } \nu;\ U_\infty} \boxed{\begin{array}{c} \dfrac{\partial \tilde{u}}{\partial x} \approx \dfrac{\partial^2 \tilde{u}}{\partial\left(\sqrt{\frac{U_\infty}{\nu}}y\right)^2} \\ \sqrt{\dfrac{U_\infty}{\nu}}y = 0 : \dfrac{\partial \tilde{u}}{\partial\left(\sqrt{\frac{U_\infty}{\nu}}y\right)} = 0 \\ \sqrt{\dfrac{U_\infty}{\nu}}y \to \infty : \tilde{u} \to 0 \\ \forall x : \dfrac{D}{2\rho U_\infty}\sqrt{\dfrac{U_\infty}{\nu}} = I = \displaystyle\int_0^\infty \tilde{u}\, d\left(\sqrt{\frac{U_\infty}{\nu}}y\right) \end{array}}$$

$$\downarrow$$

$$\tilde{u} = \tilde{u}\left(x, \sqrt{\frac{U_\infty}{\nu}}y, I\right)$$

Ecuaciones de dimensiones :

$[x] = L$

$\left[\sqrt{\frac{U_\infty}{\nu}}\,y\right] = \left(\frac{LT^{-1}}{L^2T^{-1}}\right)^{1/2} L = L^{1/2}$

$[\tilde{u}] = LT^{-1}$

$[I] = \left[\frac{D}{2\rho U_\infty}\right]\left[\sqrt{\frac{U_\infty}{\nu}}\right] = [U_\infty y]\left[\sqrt{\frac{U_\infty}{\nu}}\right] = (LT^{-1}L)\left(\frac{LT^{-1}}{L^2T^{-1}}\right)^{1/2} = L^{3/2}T^{-1}$

2 magnitudes dimensionalmente independientes

$\boxed{x, I}$

Adimensionalizamos :

$$\tilde{u} \longrightarrow \frac{\tilde{u}}{x^\alpha I^\beta} \longrightarrow \frac{LT^{-1}}{L^\alpha L^{3\beta/2}T^{-\beta}} \longrightarrow \begin{cases} \beta = 1 \\ 1 = \alpha + \frac{3\beta}{2} \end{cases} \longrightarrow \begin{cases} \alpha = -\frac{1}{2} \\ \beta = 1 \end{cases} \longrightarrow \frac{\tilde{u}\sqrt{x}}{I}$$

$$\sqrt{\frac{U_\infty}{\nu}}\,y \longrightarrow \left[\sqrt{\frac{U_\infty}{\nu}}\,y\right] = L^{1/2} \longrightarrow \text{Se adimensionaliza con } x^{1/2} \longrightarrow \sqrt{\frac{U_\infty}{\nu x}}\,y = \eta$$

Entonces :

$$\boxed{\frac{\tilde{u}\sqrt{x}}{I} = f(\eta), \text{ con } \eta = \sqrt{\frac{U_\infty}{\nu x}}\,y}$$

$$\tilde{u} = \frac{I}{\sqrt{x}} f(\eta)$$

$$\boxed{\frac{\partial \eta}{\partial x} = -\frac{\eta}{2x}} \qquad \boxed{\frac{\partial \eta}{\partial y} = \sqrt{\frac{U_\infty}{\nu x}}}$$

Si recordamos, la ECdM$_x$ quedaba reducida a : $U_\infty \frac{\partial \tilde{u}}{\partial x} \approx \nu \frac{\partial^2 \tilde{u}}{\partial y^2}$

$\boxed{\frac{\partial \tilde{u}}{\partial x}}$

$$\frac{\partial \tilde{u}}{\partial x} = \frac{\partial}{\partial x}\left[\frac{I}{\sqrt{x}} f(\eta)\right] = -\frac{1}{2}\frac{I}{x\sqrt{x}} f - \frac{I}{\sqrt{x}} f' \frac{\eta}{2x} \longrightarrow \boxed{\frac{\partial \tilde{u}}{\partial x} = -\frac{I}{2x\sqrt{x}}\left(f + \eta f'\right)}$$

$\boxed{\frac{\partial \tilde{u}}{\partial y}}$

$$\frac{\partial \tilde{u}}{\partial y} = \frac{\partial}{\partial y}\left[\frac{I}{\sqrt{x}} f(\eta)\right] = \frac{I}{\sqrt{x}} f' \sqrt{\frac{U_\infty}{\nu x}} = \frac{I}{x}\sqrt{\frac{U_\infty}{\nu}}\, f'$$

$$\boxed{\frac{\partial^2 \tilde{u}}{\partial y^2}}$$

$$\frac{\partial^2 \tilde{u}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{I}{x}\sqrt{\frac{U_\infty}{\nu}}\, f'\right) = \frac{I}{x}\sqrt{\frac{U_\infty}{\nu}}\, f''\sqrt{\frac{U_\infty}{\nu x}} \longrightarrow \boxed{\frac{\partial^2 \tilde{u}}{\partial y^2} = \frac{I}{x\sqrt{x}}\,\frac{U_\infty}{\nu}\, f''}$$

Sustituyendo :

$$-U_\infty \frac{I}{2x\sqrt{x}}\left(f + \eta f'\right) \approx \nu \frac{I}{x\sqrt{x}}\,\frac{U_\infty}{\nu}\, f''$$

$$-\frac{1}{2}\left(f + \eta f'\right) \approx f''$$

$$\boxed{f'' + \frac{\eta}{2} f' + \frac{1}{2} f \approx 0}$$

En cuanto a las condiciones de contorno :

$$y = 0 : \frac{\partial \tilde{u}}{\partial y} = 0 \longrightarrow \eta = 0 : f' = 0 \quad (1)$$

$$y \to \infty : \tilde{u} \to 0 \longrightarrow \eta \to \infty : f = 0 \quad (2)$$

Si necesitamos que $\int_0^\infty f \, d\eta$ esté acotada,

Se debe satisfacer $\eta \to \infty : f \to 0$,

Colapsando las condiciones (2) y (3)

$$\forall x : \frac{D}{2\rho U_\infty} = \int_0^\infty \tilde{u}\, dy \longrightarrow I\sqrt{\frac{\nu}{U_\infty}} = \int_0^\infty \frac{I}{\sqrt{x}} f(\eta)\, d\left(\sqrt{\frac{\nu x}{U_\infty}}\,\eta\right) \longrightarrow \boxed{\int_0^\infty f(\eta)\, d\eta = 1} \quad (3)$$

DESHACE LA HOMOGENEIDAD

"D" RESPONSABLE DE LA APARICIÓN DE LA ESTELA

Por tanto, el problema queda :

$$f'' + \frac{\eta}{2} f' + \frac{1}{2} f \approx 0$$

$$\eta = 0 : f' = 0$$

$$\forall x : \int_0^\infty f(\eta)\, d\eta = 1$$

Integración:

$$\left(f' + \frac{1}{2}\eta f \right)' \approx 0$$

$$\Big\downarrow \int$$

$$f' + \frac{1}{2}\eta f \approx C_1 \xrightarrow{\ \eta = 0\ :\ f' = 0\ } C_1 \approx 0$$

$$\Big\downarrow$$

$$f' + \frac{1}{2}\eta f \approx 0$$

$$\Big\downarrow \int$$

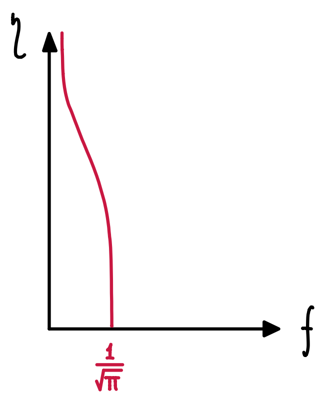
$$\int f'/f \; df \approx -\frac{1}{2}\int \eta \, d\eta$$

$$\ln f + k_1 \approx -\frac{\eta^2}{4} + k_2$$

$$f \approx C_2 \exp\left(-\frac{\eta^2}{4}\right) \xrightarrow{\ \int_0^\infty f \, d\eta = 1\ } C_2 \approx \frac{1}{\int_0^\infty \exp\left(-\frac{\eta^2}{4}\right) d\eta} = \frac{1}{\sqrt{\pi}}$$

$$\Big\downarrow$$

$$\boxed{f(\eta) \approx \frac{1}{\sqrt{\pi}} \exp\left(-\frac{\eta^2}{4}\right)}$$



En cuanto a "u":

$$u = U_\infty - \tilde{u} = U_\infty - \frac{1}{\sqrt{x}} \frac{D}{2\rho U_\infty} \sqrt{\frac{\nu}{U_\infty}} \frac{1}{\sqrt{\pi}} \exp\left(-\frac{\eta^2}{4}\right)$$

$$\boxed{u = U_\infty - \frac{\frac{D}{\rho}}{2\sqrt{\pi \nu x U_\infty}} \exp\left(-\frac{U_\infty}{4\nu x} y^2\right)}$$

Nota: $\boxed{\tilde{u}\big|_{y=0} \sim x^{-1/2}}$

El perfil de velocidades se va uniformizando al alejarnos del obstáculo.