Métodos Matemáticos de Bioingeniería
Grado en Ingeniería Biomédica
Lecture 10

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Outline

1. Directional Derivatives and the Gradient
   - Review
   - Directional Derivatives
Directional Derivatives and the Gradient

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   - Directional Derivatives
Recalling Geometric Interpretation of Partial Derivatives

Geometrically it is the slope at the point \((a, b, f(a, b))\) of the curve obtained by intersecting:

- The surface \(z = f(x, y)\) with the plane \(y = b\).
Partial Derivatives as Directional Derivatives

Consider an alternative geometric way to view $\frac{\partial f}{\partial x}(a, b)$:

- It can be viewed as the rate of change of $f$ as we move “infinitesimally” from $a = (a, b)$ in the $i$-direction.
Partial Derivatives as Directional Derivatives

- By the definition of the partial derivative:

\[
\frac{\partial f}{\partial x}(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h} = \lim_{h \to 0} \frac{f((a, b) + (h, 0)) - f(a, b)}{h} = \lim_{h \to 0} \frac{f((a, b) + h(1, 0)) - f(a, b)}{h}
\]

- Similarly, we have,

\[
\frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a + hj) - f(a)}{h}
\]
General Directional Derivatives

\[
\frac{\partial f}{\partial x}(a, b) = \lim_{h \to 0} \frac{f(a + hi) - f(a)}{h} \\
\frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a + hj) - f(a)}{h}
\]

- Partial derivatives are special cases of a more general type of derivative,
- Suppose \( \mathbf{v} \) is any unit vector in \( \mathbb{R}^2 \) and consider the quantity,

\[
\lim_{h \to 0} \frac{f(a + hv) - f(a)}{h}
\]

It is the rate of change of \( f \) as we move (infinitesimally) from \( a = (a, b) \) in the direction specified by \( \mathbf{v} = (A, B) = Ai + Bj \).
### General Directional Derivatives

\[
\lim_{h \to 0} \frac{f(a + hv) - f(a)}{h}, \quad \mathbf{v} = (A, B) = A\mathbf{i} + B\mathbf{j}
\]

- It is also the slope of the curve obtained as the intersection of
  - The surface \( z = f(x, y) \), with the vertical plane,

\[
B(x - a) - A(y - b) = 0
\]
Outline

1. Directional Derivatives and the Gradient
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Directional Derivatives and the Gradient

Directional Derivatives

Definition 6.1

Let $X$ be an open in $\mathbb{R}^n$ and $a \in X$.

Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function.

If $v \in \mathbb{R}^n$ is any unit vector, then the directional derivative of $f$ at $a$ in the direction of $v$ is:

\[
D_v f(a) = \lim_{h \to 0} \frac{f(a + hv) - f(a)}{h}
\]

Provided that this limit exists.
Example 1

- Suppose,
  
  \[ f(x, y) = x^2 - 3xy + 2x - 5y \]

- Then, if \( \mathbf{v} = (v, w) \in \mathbb{R}^2 \) is any unit vector, it follows that

  \[
  D_v f(0, 0) = \lim_{h \to 0} \frac{f((0, 0) + h(v, w)) - f(0, 0)}{h}
  \]

  \[
  = \lim_{h \to 0} \frac{h^2 v^2 - 3h^2 vw + 2hv - 5hw}{h}
  = \lim_{h \to 0} (hv^2 - 3hv w + 2v - 5w)
  = 2v - 5w
  \]

- Thus
  
  - The rate of change of \( f \) is \( 2v - 5w \) if we move from the origin in the direction given by \( \mathbf{v} \).
  - The rate of change is zero if \( \mathbf{v} = (5/\sqrt{29}, 2/\sqrt{29}) \) or \( \mathbf{v} = (-5/\sqrt{29}, -2/\sqrt{29}) \).
Theorem 6.2

Let $X$ be an open in $\mathbb{R}^n$

Suppose $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at $a \in X$

Then the directional derivative $D_v f(a)$ exists for all directions (unit vectors) $v \in \mathbb{R}^n$ and, moreover, we have

$$D_v f(a) = \nabla f(a) \cdot v$$

Example 2

Recall function in Example 1

$$f(x, y) = x^2 - 3xy + 2x - 5y$$

For any unit vector $v = vi + wj \in \mathbb{R}^2$

$$D_v f(0, 0) = \nabla f(0, 0) \cdot v = (f_x(0, 0)i + f_y(0, 0)j) \cdot (vi + wj)$$

$$= (2i - 5j) \cdot (vi + wj) = 2v - 5w$$
Theorem 6.2

- Let $X$ be an open in $\mathbb{R}^n$
- Suppose $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at $a \in X$
- Then the directional derivative $D_v f(a)$ exists for all directions (unit vectors) $v \in \mathbb{R}^n$ and, moreover, we have

$$D_v f(a) = \nabla f(a) \cdot v$$

Remark

- The converse of Theorem 6.2 does not hold

A function may have directional derivatives in all directions at a point yet fail to be differentiable.
Theorem 6.3: Steepest Ascent

The directional derivative $D_u f(a)$ is:

- Maximized, with respect to direction, when $u$ points in the same direction as $\nabla f(a)$, and
- Minimized, when $u$ points in the opposite direction.

Furthermore, the maximum and minimum values of $D_u f(a)$ are:

$$\|\nabla f(a)\| \quad \text{and} \quad -\|\nabla f(a)\|,$$

respectively.
Example 4

- Imagine you are traveling in space near the planet Nilrebo.
- Suppose that one of your spaceship’s instruments measures the external atmospheric pressure on your ship.
- This external atmospheric pressure is measured as a function $f(x, y, z)$ of position.
- Assume that this function $f$ is differentiable.
- Then Theorem 6.2 can be applied.
- If you travel from point $a = (a, b, c)$ in the direction of the (unit) vector $\mathbf{u} = ui + vj + wk$, the rate of change of pressure is given by

$$D_{\mathbf{u}}f(a) = \nabla f(a) \cdot \mathbf{u}$$

In what direction is the pressure increasing the most?
Example 4

- In particular, suppose the pressure function on Nilrebo is
  \[ f(x, y, z) = 5x^2 + 7y^4 + x^2z^2 \text{ atm} \]

- Assume the origin is located at the center of Nilrebo and distance units are measured in thousands of kilometers

- The rate of change of pressure at \((1, -1, 2)\) in the direction of \(\mathbf{i} + \mathbf{j} + \mathbf{k}\) may be calculated as

  \[ \nabla f(1, -1, 2) \cdot \mathbf{u}, \quad \text{where } \mathbf{u} = (\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3} \] (normalized)

- Using Theorem 6.2

  \[
  D_\mathbf{u} f(1, -1, 2) = \nabla f(1, -1, 2) \cdot \mathbf{u} = 18\mathbf{i} - 28\mathbf{j} + 4\mathbf{k} \cdot \frac{(\mathbf{i} + \mathbf{j} + \mathbf{k})}{\sqrt{3}} \\
  = \frac{18 - 28 + 4}{\sqrt{3}} = -2\sqrt{3} \text{ atm/Mm}
  \]
Example 4

Suppose the pressure function on Nilrebo is

\[ f(x, y, z) = 5x^2 + 7y^4 + x^2z^2 \text{ atm} \]

Assume the origin is located at the center of Nilrebo and distance units are measured in thousands of kilometers.

The rate of change of pressure at \((1, -1, 2)\) in the direction of \(\mathbf{i} + \mathbf{j} + \mathbf{k}\) may be calculated as

\[ \nabla f(1, -1, 2) \cdot \mathbf{u}, \quad \text{where } \mathbf{u} = (\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3} \text{ (normalized)} \]

Using Theorem 6.3, the pressure will increase most rapidly in the direction of \(\nabla f(1, -1, 2)\), that is, in the direction

\[ \frac{18\mathbf{i} - 28\mathbf{j} + 4\mathbf{k}}{\|18\mathbf{i} - 28\mathbf{j} + 4\mathbf{k}\|} = \frac{9\mathbf{i} - 14\mathbf{j} + 2\mathbf{k}}{\sqrt{281}} \]

The rate of this increase is \(\|\nabla f(1, -1, 2)\| = 2\sqrt{281} \text{ atm/Mm}\)
Steepest Ascent

- **Theorem 6.3** is independent of the dimension. It applies to functions $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ for any $n \geq 2$.

- For $n = 2$, there is another geometric interpretation of Theorem 6.3.

- Suppose you are mountain climbing on the surface $z = f(x, y)$. 

![Mountain Climbing Illustration](image-url)
Steepest Ascent

- Suppose you are mountain climbing on the surface $z = f(x, y)$

- Think of the value of $f$ as the height of the mountain above (or below) sea level
Steepest Ascent

- Suppose you are equipped with a map and compass, which supply information in the xy-plane only
Assume you are at the point on the mountain with \( xy \)-coordinates (map coordinates) \((a, b)\)

To climb the mountain faster, **Theorem 6.3** says that you should move in the direction parallel to the gradient \( \nabla f(a, b) \)
Similarly, you should move in the direction parallel to $-\nabla f(a, b)$ in order to descend most rapidly.

Moreover, the slope of your ascent or descent in these cases is $\|\nabla f(a, b)\|$.
Steepest Ascent

\[ \nabla f(a, b) \] is a vector in \( \mathbb{R}^2 \) that gives the optimal north-south, east-west direction of travel.