Métodos Matemáticos de Bioingeniería

Grado en Ingeniería Biomédica

Lecture 13

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Outline

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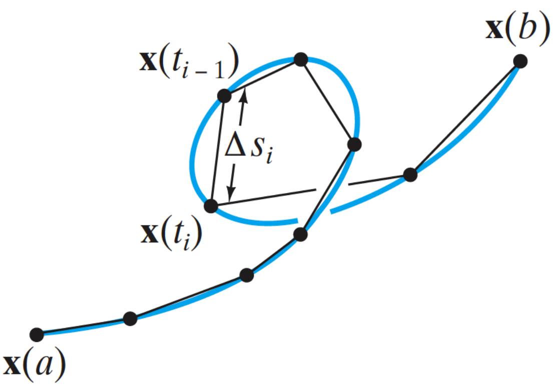
Differential Geometry

- We focus on the study of parametrized curves in $\mathbb{R}^3$.
- We consider how to measure geometric properties as length and curvature.
- A deeper insight can be obtained by defining three mutually perpendicular unit vectors: the so-called moving frame.
- This study takes us briefly into the branch of mathematics called differential geometry.
- This area uses calculus and analysis to understand the geometry of
  - Curves
  - Surfaces, and
  - Manifolds

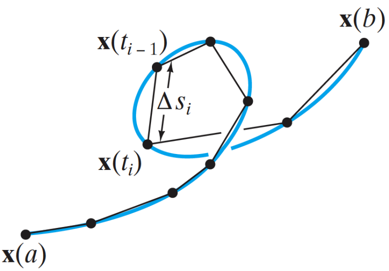
Outline

Approximating the length of a C^1 path

- Let $\mathbf{x} : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^3$ be a C^1 path in $\mathbb{R}^3$
- We can approximate the length L of $\mathbf{x}$ as:



Approximating the length of a C^1 path

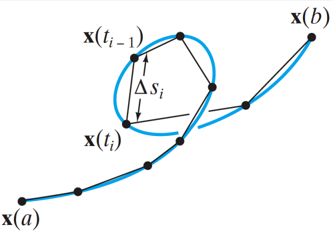


- First, partition the interval $[a, b]$ into n subintervals.
- Choose numbers $t_0, t_1, \ldots, t_n$ such that,

$$a = t_0 < t_1 < \cdots < t_n = b$$

- For $i = 1, \ldots, n$, we let Δs_i denote the distance between the points $\mathbf{x}(t_{i-1})$ and $\mathbf{x}(t_i)$ on the path.

Approximating the length of a C^1 path

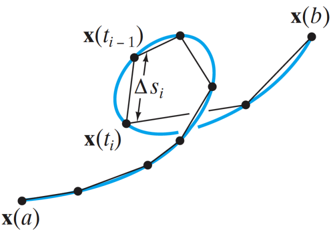


$$a = t_0 < t_1 < \cdots < t_n = b$$
$$\Delta s_i = \text{distance between } \mathbf{x}(t_{i-1}) \text{ and } \mathbf{x}(t_i)$$

- Then, we can approximate L

$$L \approx \sum_{i=1}^{n} \Delta s_i$$

Approximating the length of a C^1 path



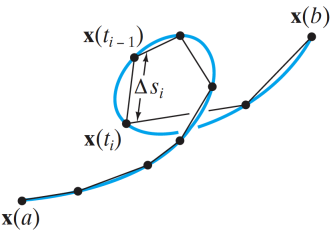
$$a = t_0 < t_1 < \cdots < t_n = b$$

$$\Delta s_i = \text{distance between } \mathbf{x}(t_{i-1}) \text{ and } \mathbf{x}(t_i)\ , \quad L \approx \sum_{i=1}^{n} \Delta s_i$$

- Since $\mathbf{x}(t) = (x(t), y(t), z(t))$ and using the Pythagorean theorem:

$$\Delta s_i = \sqrt{\Delta x_i^2 + \Delta y_i^2 + \Delta z_i^2}$$

Approximating the length of a C^1 path



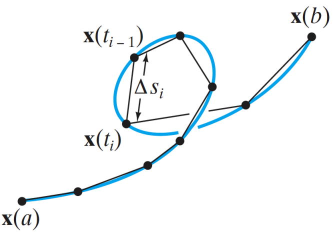
$$a = t_0 < t_1 < \cdots < t_n = b$$

$$\Delta s_i = \sqrt{\Delta x_i^2 + \Delta y_i^2 + \Delta z_i^2}, \quad L \approx \sum_{i=1}^{n} \Delta s_i$$

- Then, we define the length L of $\mathbf{x}$ to be

$$L = \lim_{\max \Delta t_i \to 0} \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2 + \Delta z_i^2}$$

Approximating the length of a C^1 path

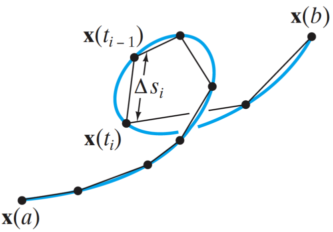


$$
\begin{aligned}
a &= t_0 < t_1 < \cdots < t_n = b \\
L &= \lim_{\max \Delta t_i \to 0} \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2 + \Delta z_i^2}
\end{aligned}
$$

- We can rewrite this equation as an integral:

$$
L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2 dt}
$$

Approximating the length of a C^1 path



$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt$$

- Note that the integrand is precisely the speed of the path.

$$\|\mathbf{x}'(t)\|$$

- Speed measures the rate of distance traveled per unit time.
- So, it make sense that integrating the speed over the elapsed time interval should give the total distance traveled.

Definition 2.1: Length of a Path in $\mathbb{R}^n$

- The length $L(\mathbf{x})$ of a C^1 path $\mathbf{x} : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ is found by integrating its speed

$$L(\mathbf{x}) = \int_a^b \|\mathbf{x}'(t)\| dt$$

Example 1

- We compute the length of the path,

$$\mathbf{x} : [0, 2\pi] \to \mathbb{R}^2, \quad \mathbf{x}(t) = (a\cos t, a\sin t), \; a > 0$$

- We have,

$$\begin{aligned} \mathbf{x}'(t) &= -a\sin t\mathbf{i} + a\cos t\mathbf{j} \\ \|\mathbf{x}'(t)\| &= \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a \end{aligned}$$

Definition 2.1: Length of a Path in $\mathbb{R}^n$

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Example 1

$$\begin{aligned} \mathbf{x}'(t) &= -a \sin t\mathbf{i} + a \cos t\mathbf{j} \\ \|\mathbf{x}'(t)\| &= \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} = a \end{aligned}$$

- Thus, Definition 2.1 gives

$$L(\mathbf{x}) = \int_a^b \|\mathbf{x}'(t)\| dt = \int_0^{2\pi} a dt = 2\pi a$$

Definition 2.1: Length of a Path in $\mathbb{R}^n$

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Example 1

$$\begin{aligned} \mathbf{x}'(t) &= -a \sin t\mathbf{i} + a \cos t\mathbf{j} \\ \|\mathbf{x}'(t)\| &= \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} = a \\ L(\mathbf{x}) &= 2\pi a \end{aligned}$$

- Notice that the path $\mathbf{x}$ traces a circle of radius a once.
- The length integral works out to be the circumference of the circle, as it should.

Definition 2.1: Length of a Path in $\mathbb{R}^n$

- The length $L(\mathbf{x})$ of a C^1 path $\mathbf{x} : [a, b] \subseteq \mathbb{R} \to \mathbb{R}^n$ is found by integrating its speed

$$L(\mathbf{x}) = \int_a^b \|\mathbf{x}'(t)\| dt$$

Example 2

- Consider the helix

$$\mathbf{x} : [0, 2\pi] \to \mathbb{R}^3, \quad \mathbf{x}(t) = (a \cos t, a \sin t, bt), \ 0 \leq t \leq 2\pi$$

- We have

$$\begin{aligned} \mathbf{x}'(t) &= -a \sin t\mathbf{i} + a \cos t\mathbf{j} + b\mathbf{k} \\ \|\mathbf{x}'(t)\| &= \sqrt{a^2 + b^2} \\ L(\mathbf{x}) &= \int_a^b \|\mathbf{x}'(t)\| dt = \int_0^{2\pi} \sqrt{a^2 + b^2} dt = 2\pi\sqrt{a^2 + b^2} \end{aligned}$$

Definition 2.1: Length of a Path in $\mathbb{R}^n$

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$$L(\mathbf{x}) = \int_a^b \|\mathbf{x}'(t)\| dt$$

Example 2

$$\mathbf{x} : [0, 2\pi] \to \mathbb{R}^3, \quad \mathbf{x}(t) = (a\cos t, a\sin t, bt), \; 0 \leq t \leq 2\pi$$

$$L(\mathbf{x}) = 2\pi\sqrt{a^2 + b^2}$$

- When $b = 0$, the helix reverts to a circle and the length integral agrees with the previous example

Outline

Reparametrization of a Path

- The calculation of the length of a path provides a way to reparametrize the path
- This reparametrization uses a parameter that depends solely on the geometry of the curve traced by the path.

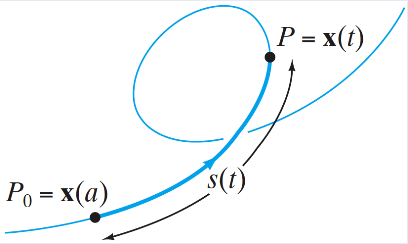
It does not depend on
the way in which the curve is traced

- This parameter is called the arclength parameter.

Reparametrization of a Path

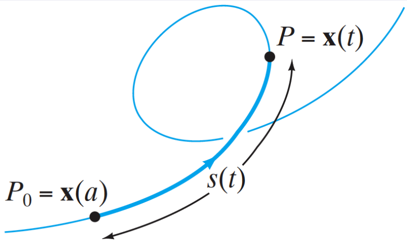
- Let $\mathbf{x}$ be any C^1 path and assume that the velocity $\mathbf{x}' \neq \mathbf{0}$.
- Fix a point P_0 on the path and let a be such that $\mathbf{x}(a) = P_0$.
- We define a one-variable function s of the given parameter t
- This function s measures the length of the path from P_0 to any other (moving) point P by:

$$s(t) = \int_a^t \|\mathbf{x}'(\tau)\| d\tau$$



Reparametrization of a Path

$$s(t) = \int_a^t \|\mathbf{x}'(\tau)\| d\tau$$

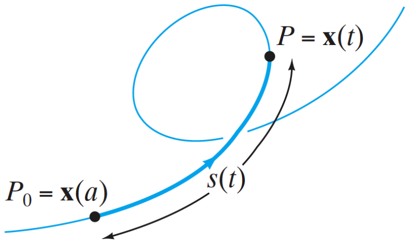


- From the former formula and from the fundamental theorem of calculus

$$\frac{ds}{dt} = \frac{d}{dt}\int_a^t \|\mathbf{x}'(\tau)\| d\tau = \|\mathbf{x}'(t)\| = \text{speed}$$

Reparametrization of a Path

$$s(t) = \int_a^t \|\mathbf{x}'(\tau)\| d\tau, \quad \frac{ds}{dt} = \|\mathbf{x}'(t)\| = \text{speed}$$



- Since we have assumed that $\mathbf{x}' \neq \mathbf{0}$, it follows that ds/dt is nonzero
- In fact, s is an invertible function

It is at least theoretically possible
to solve the equation $s = s(t)$ for t in terms of s.

Example 3

- Consider the helix

$$\mathbf{x} : [0, 2\pi] \to \mathbb{R}^3, \quad \mathbf{x}(t) = (a\cos t, a\sin t, bt),\ 0 \le t \le 2\pi$$

- Let choose the "base point" P_0 to be $\mathbf{x}(0) = (a, 0, 0)$
- Then we have

$$s(t) = \int_0^t \|\mathbf{x}'(\tau)\| d\tau = \int_0^t \sqrt{a^2 + b^2} d\tau = \sqrt{a^2 + b^2}\, t$$

- So that

$$s = \sqrt{a^2 + b^2}\, t \Rightarrow t = \frac{s}{\sqrt{a^2 + b^2}}$$

This reparametrization
just rescales the time variable

Example 3

- Consider the helix

$$\mathbf{x} : [0, 2\pi] \to \mathbb{R}^3, \quad \mathbf{x}(t) = (a\cos t, a\sin t, bt), \; 0 \le t \le 2\pi$$

- Let choose the "base point" P_0 to be $\mathbf{x}(0) = (a, 0, 0)$
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$$s(t) = \int_0^t \|\mathbf{x}'(\tau)\| d\tau = \int_0^t \sqrt{a^2 + b^2} d\tau = \sqrt{a^2 + b^2}\, t$$

- So that

$$s = \sqrt{a^2 + b^2}\, t \Rightarrow t = \frac{s}{\sqrt{a^2 + b^2}}$$

- Hence, we can rewrite the helical path as

$$\mathbf{x}(s) = \left(a\cos\left(\frac{s}{\sqrt{a^2 + b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2 + b^2}}\right), \frac{bs}{\sqrt{a^2 + b^2}} \right)$$

Interpretation of the Arclength Parametrization

- The arclength parameter s is an intrinsic parameter.
- It depends only on how the curve itself bends.
- It does not depend on how fast (or slowly) the curve is traced.
- Using the chain rule

$$\mathbf{x}'(t) = \frac{d(\mathbf{x}(s) \circ s(t))}{dt} = \mathbf{x}'(s)\frac{ds}{dt} = \mathbf{x}'(s)\|\mathbf{x}'(t)\|$$

- Since $\mathbf{x}'(t) \neq \mathbf{0}$,

$$\mathbf{x}'(s) = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}$$

- Therefore, $\mathbf{x}'(s)$ is precisely the normalization of the original velocity vector, and so it is a unit vector.
- Hence, the reparametrized path $\mathbf{x}(s)$ has unit speed, regardless of the speed of the original path $\mathbf{x}(t)$.

Outline

Definition 2.2

- Let $\mathbf{x} : I \subseteq \mathbb{R} \to \mathbb{R}^3$ be a C^3 path and assume that $\mathbf{x}' \neq \mathbf{0}$
- The unit tangent vector $\mathbf{T}$ of the path $\mathbf{x}$ is the normalization of the velocity vector,

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}$$

Remarks

- $\mathbf{T}$ is undefined when the speed of the path is zero.
- $\mathbf{T}$ is $d\mathbf{x}/ds$, where s is the arclength parameter.

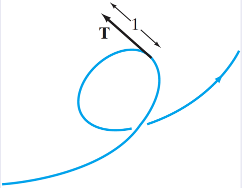
Definition 2.2

- Let $\mathbf{x} : I \subseteq \mathbb{R} \to \mathbb{R}^3$ be a C^3 path and assume that $\mathbf{x}' \neq \mathbf{0}$
- The unit tangent vector $\mathbf{T}$ of the path $\mathbf{x}$ is the normalization of the velocity vector,

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}$$

Remarks

- Geometrically, $\mathbf{T}$ is the tangent vector of unit length that points in the direction of increasing arclength



Example 5

- Consider the helix

$$\mathbf{x}(t) = (a\cos t, a\sin t, bt)$$

- Then

$$\mathbf{T}(t) = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} = \frac{-a\sin t\mathbf{i} + a\cos t\mathbf{j} + b\mathbf{k}}{\sqrt{a^2+b^2}}$$

- On the other hand, if we parametrize the helix using arclength

$$\mathbf{x}(s) = \left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{bs}{\sqrt{a^2+b^2}}\right)$$

- Then

$$\mathbf{T}(s) = \mathbf{x}'(s) = \frac{-a}{\sqrt{a^2+b^2}}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right)\mathbf{i} + \frac{a}{\sqrt{a^2+b^2}}\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right)\mathbf{j}$$
$$+ \quad \frac{b}{\sqrt{a^2+b^2}}\mathbf{k} \quad (\text{Recall } s = \sqrt{a^2+b^2}\,t)$$

Proposition 2.3

- Assume that the path $\mathbf{x}$ always has nonzero speed.
- Then
  1. $d\mathbf{T}/dt$ is perpendicular to $\mathbf{T}$ for all t in I (the domain of the path $\mathbf{x}$).
  2. $\|d\mathbf{T}/dt\| \,|_{t=t_0}$ equals the angular rate of change (as t increases) of the direction of $\mathbf{T}$ when $t = t_0$.

Remark

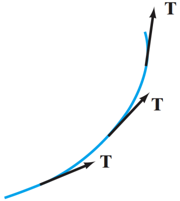
- Using the unit tangent vector, we can define a quantity that measures how much a path bends as we travel along it.
- Part 2 of Proposition 2.3 provides a precise way of measuring the bending of a path.

Definition 2.4

The curvature κ of a path $\mathbf{x}$ in $\mathbb{R}^3$ is the angular rate of change of the direction of $\mathbf{T}$ per unit change in distance along the path.

Remarks

- Note we are taking the rate of change of $\mathbf{T}$ per unit change in distance.
- The reason is that we want the curvature κ to be an intrinsic quantity



Small curvature κ

Definition 2.4

- The curvature κ of a path $\mathbf{x}$ in $\mathbb{R}^3$ is the angular rate of change of the direction of $\mathbf{T}$ per unit change in distance along the path.

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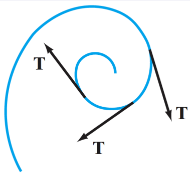

Large curvature κ

Definition 2.4

- The curvature κ of a path $\mathbf{x}$ in $\mathbb{R}^3$ is the angular rate of change of the direction of $\mathbf{T}$ per unit change in distance along the path.

Remarks

- Considering Definition 2.4, Part 2 of Proposition 2.3 and using the chain rule

$$\kappa(t) = \frac{\|d\mathbf{T}/dt\|}{ds/dt} = \left\| \frac{d\mathbf{T}}{ds} \right\|$$

  - $\|d\mathbf{T}/dt\|$ measures the angular rate of change of the direction of $\mathbf{T}$ per unit change in parameter.
  - ds/dt is the rate of change of distance per unit change in parameter.

Example 6

- Consider the circle,

$$\mathbf{x}(t) = (a\cos t, a\sin t),\ 0 \le t < 2\pi$$

- Then,

$$\begin{aligned} \mathbf{x}'(t) &= -a\sin t\mathbf{i} + a\cos t\mathbf{j} \\ \|\mathbf{x}'(t)\| &= \frac{ds}{dt} = a \end{aligned}$$

- So that,

$$\mathbf{T}(t) = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

- Hence,

$$\kappa(t) = \frac{\|d\mathbf{T}/dt\|}{ds/dt} = \frac{1}{a}\|-\cos t\mathbf{i} - \sin t\mathbf{j}\| = \frac{1}{a}$$

Example 6

- Consider the circle,

$$\begin{aligned} \mathbf{x}(t) &= (a\cos t, a\sin t),\ 0 \leq t < 2\pi \\ \kappa(t) &= \frac{\|d\mathbf{T}/dt\|}{ds/dt} = \frac{1}{a}\| - \cos t\mathbf{i} - \sin t\mathbf{j}\| = \frac{1}{a} \end{aligned}$$

- The curvature of a circle is always constant.
- Its value equals the reciprocal of the radius.

Therefore, the smaller the circle,
the greater the curvature

Example 7

- Let $\mathbf{a}$ and $\mathbf{b}$ be constant vectors in $\mathbb{R}^3$ and $\mathbf{a} \neq \mathbf{0}$.
- Consider the path,

$$\mathbf{x}(t) = \mathbf{a}t + \mathbf{b}$$

- This path traces a line.
- Then,

$$\mathbf{x}'(t) = \mathbf{a} \quad \text{and} \quad \|\mathbf{x}'(t)\| = \frac{ds}{dt} = \|\mathbf{a}\|$$

- Hence,

$$\mathbf{T}(t) = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

- Therefore $\mathbf{T}(t)$ is a constant vector.

Example 7

- Let $\mathbf{a}$ and $\mathbf{b}$ be constant vectors in $\mathbb{R}^3$ and $\mathbf{a} \neq \mathbf{0}$
- Consider the path

$$\mathbf{x}(t) = \mathbf{a}t + \mathbf{b}$$

- Therefore $\mathbf{T}(t)$ is a constant vector.
- Thus,

$$\mathbf{T}'(t) \equiv \mathbf{0} \Rightarrow \kappa = 0$$

This result agrees with the intuitive fact that
a line doesn't curve

Example 8

- Consider the helix

$$\mathbf{x}(t) = (a\cos t, a\sin t, bt)$$

- We have already seen that

$$\frac{ds}{dt} = \sqrt{a^2 + b^2} \quad \text{and} \quad \mathbf{T}(t) = \frac{-a\sin t\mathbf{i} + a\cos t\mathbf{j} + b\mathbf{k}}{\sqrt{a^2 + b^2}}$$

- Thus

$$\kappa(t) = \frac{\|d\mathbf{T}/dt\|}{ds/dt} = \frac{1}{\sqrt{a^2+b^2}} \left\| \frac{-a\cos t\mathbf{i} - a\sin t\mathbf{j}}{\sqrt{a^2+b^2}} \right\| = \frac{a}{a^2+b^2}$$

The curvature of the helix
is constant, just like the circle

κ in terms on nonintrinsic quantities

Let $c(t)$ be a smooth curve. Another interesting formula for the curvature is

$$\kappa(t) = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}$$

where $\mathbf{v}(\mathrm{t})$ and $\mathbf{a}(\mathrm{t})$ are the velocity and acceleration vector respectively.

Try to compute the curvature of $f(x) = x^2$ and $f(x) = sin(x)$. Does it makes sense?