Métodos Matemáticos de Bioingeniería
Grado en Ingeniería Biomédica
Lecture 20

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Outline

1. Scalar and Vector Line Integrals
   - Scalar line integral
   - Vector line integral
   - Differential form of the line integral
   - Effect of reparametrization
   - Closed and simples curves

2. Green’s Theorem
   - Definition
   - Examples
1 Scalar and Vector Line Integrals
   - Scalar line integral
   - Vector line integral
   - Differential form of the line integral
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2 Green’s Theorem
   - Definition
   - Examples
Scalar and Vector Line Integrals

Scalar line integral

Scalar Line Integral as a limit of a Riemann sum

- Let \( \mathbf{x} : [a, b] \to \mathbb{R}^3 \) be a path of class \( C^1 \)
- Let \( f : X \subseteq \mathbb{R}^3 \to \mathbb{R} \) be a continuous function
- Suppose that domain \( X \) contains the image of \( \mathbf{x} \), so that the composite \( f(\mathbf{x}(t)) \) is defined
- As with every other integral, the scalar line integral is a limit of appropriate Riemann sums
- Consider a partition of \([a, b]\)

\[
a = t_0 < t_1 < \cdots < t_k < \cdots < t_n = b
\]
Scalar Line Integral as a limit of a Riemann sum

\[ a = t_0 < t_1 < \cdots < t_k < \cdots < t_n = b \]

- Let us think of
  - The image of the path \( \mathbf{x} \) as representing an idealized wire in space
  - \( f(\mathbf{x}(t)) \) as the electrical charge density of the wire

Then, the Riemann sum approximates the total charge of the wire

\[
\text{Total charge} = \lim_{\Delta t_k \to 0} \sum_{k=1}^{n} f(\mathbf{x}(t_k^*)) \Delta s_k
\]
Definition 1.1: Scalar Line Integral

- The **scalar line integral** of $f$ along the $C^1$ path $x$ is

\[ \int_a^b f(x(t)) \| x'(t) \| \, dt \]

- We denote this integral

\[ \int_x f \, ds \]

Remarks

- The line integral represents a sum of values of $f$ along $x$, times “infinitesimal” pieces of **arclength** of $x$
Remarks

- **Definition 1.1** can be made for arbitrary $n$, that is, for functions $f$ defined on domains in $\mathbb{R}^n$ for arbitrary $n$.

Remarks

- We can still define the scalar line integral if:
  - $x$ is not of class $C^1$, but only "piecewise" $C^1$
  - $f(x(t))$ is only piecewise continuous
Example 1

- Let $f(x, y, z) = xy + z$ and $\mathbf{x} : [0, 2\pi] \rightarrow \mathbb{R}^3$ be the helix
  
  $$\mathbf{x}(t) = (\cos t, \sin t, t)$$

- We compute
  
  $$\int_{\mathbf{x}} f \, ds = \int_{0}^{2\pi} f(\mathbf{x}(t)) \|\mathbf{x}'(t)\| \, dt$$

- First, from the double-angle formula
  
  $$f(\mathbf{x}(t)) = \cos t \sin t + t = \frac{1}{2} \sin 2t + t$$

  $$\mathbf{x}'(t) = (-\sin t, \cos t, 1)$$

  $$\|\mathbf{x}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$
Example 1

\[ f(x, y, z) = xy + z \quad \text{and} \quad x(t) = (\cos t, \sin t, t) \]

\[ f(x(t)) = \cos t \sin t + t = \frac{1}{2} \sin 2t + t \]

\[ x'(t) = (-\sin t, \cos t, 1), \quad \|x'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2} \]

Thus

\[
\int_{x} f \, ds = \int_{0}^{2\pi} f(x(t))\|x'(t)\| \, dt = \int_{0}^{2\pi} \left( \frac{1}{2} \sin 2t + t \right) \sqrt{2} \, dt \\
= \sqrt{2} \int_{0}^{2\pi} \left( \frac{1}{2} \sin 2t + t \right) \, dt = \sqrt{2} \left[ -\frac{1}{4} \cos 2t + \frac{1}{2} t^2 \right]_{0}^{2\pi} \\
= \sqrt{2} \left( \left( -\frac{1}{4} + 2\pi^2 \right) - \left( -\frac{1}{4} + 0 \right) \right) = 2\sqrt{2}\pi^2
\]
Example 2

Let \( f(x, y) = y - x \) and let \( \mathbf{x} : [0, 3] \rightarrow \mathbb{R}^2 \) be the planar path

\[
\mathbf{x}(t) = \begin{cases} 
(2t, t) & \text{if } 0 \leq t \leq 1 \\
(t + 1, 5 - 4t) & \text{if } 1 < t \leq 3
\end{cases}
\]

Hence, \( \mathbf{x} \) is piecewise \( C^1 \) path

The two path segments defined for \( t \) in \([0, 1]\) and for \( t \) in \([1, 3]\) are each of class \( C^1 \)
Example 2

Let \( f(x, y) = y - x \) and let \( \mathbf{x} : [0, 3] \to \mathbb{R}^2 \) be the planar path

\[
\mathbf{x}(t) = \begin{cases} 
(2t, t) & \text{if } 0 \leq t \leq 1 \\
(t + 1, 5 - 4t) & \text{if } 1 < t \leq 3
\end{cases}
\]

Thus

\[
\int_{\mathbf{x}} f \, ds = \int_{\mathbf{x}_1} f \, ds + \int_{\mathbf{x}_2} f \, ds
\]
Example 2

Let \( f(x, y) = y - x \) and let \( x : [0, 3] \rightarrow \mathbb{R}^2 \) be the planar path

\[
x(t) = \begin{cases} 
(2t, t) & \text{if } 0 \leq t \leq 1 \\
(t + 1, 5 - 4t) & \text{if } 1 < t \leq 3
\end{cases}
\]

Thus

\[
\int_x f \ ds = \int_{x_1} f \ ds + \int_{x_2} f \ ds
\]

where

- \( x_1(t) = (2t, t) \) for \( 0 \leq t \leq 1 \)
- \( x_2(t) = (t + 1, 5 - 4t) \) for \( 1 < t \leq 3 \)

It is easy to see that

\[
\|x_1'(t)\| = \sqrt{5} \quad \text{and} \quad \|x_2'(t)\| = \sqrt{17}
\]
Example 2

Let \( f(x, y) = y - x \) and let \( x : [0, 3] \to \mathbb{R}^2 \) be the planar path

\[
x(t) = \begin{cases} 
(2t, t) & \text{if } 0 \leq t \leq 1 \\
(t + 1, 5 - 4t) & \text{if } 1 < t \leq 3 
\end{cases}
\]

\|x_1'(t)\| = \sqrt{5} \text{ and } \|x_2'(t)\| = \sqrt{17}

Thus

\[
\int_{x_1} f \, ds = \int_0^1 f(x_1(t)) \|x_1'(t)\| \, dt = \int_0^1 (t - 2t) \cdot \sqrt{5} \, dt = -\frac{\sqrt{5}}{2} t^2 \bigg|_0^1 = -\frac{\sqrt{5}}{2}
\]

\[
\int_{x_2} f \, ds = \int_1^3 f(x_2(t)) \|x_2'(t)\| \, dt = \int_1^3 ((5 - 4t) - (t + 1)) \cdot \sqrt{17} \, dt
\]

\[
= \sqrt{17} \left( 4t - \frac{5}{2} t^2 \right) \bigg|_1^3 = -12\sqrt{17}
\]
Geometric Interpretation of Scalar Line Integrals

- Let \( f(x, y) = 2 + x^2 y \) and let \( x : [0, \pi] \rightarrow \mathbb{R}^2 \) be the planar path

\[
x(t) = (\cos t, \sin t), \quad 0 \leq t \leq \pi
\]

- Then

\[
f(x(t)) = f(x(t), y(t)) = 2 + \cos^2 t \sin t
\]

- The line integral of \( f \) along \( x \) is the area of the “fence” whose
  - Path is governed by \( x \)
  - Height is governed by \( f \)
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2. Green’s Theorem
   - Definition
   - Examples
Definition 1.2

- Let \( x : [a, b] \rightarrow \mathbb{R}^n \) be a path of class \( C^1 \)
- Let \( F : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a vector field
- Suppose that \( X \) contains the image of \( x \) and assume that \( F \) varies continuously along \( x \)
- The vector line integral of \( F \) along \( x : [a, b] \rightarrow \mathbb{R}^n \), is

\[
\int_x F \cdot ds = \int_a^b F(x(t)) \cdot x'(t) dt
\]

Remarks

- As with scalar line integrals, we may define the vector line integrals when \( x \) is a piecewise \( C^1 \) path
- We just need to break up the integral in a suitable manner
Example 3

- Let $\mathbf{F}$ be the radial vector field on $\mathbb{R}^3$ given by
  \[ \mathbf{F} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \]
- Let $\mathbf{x} : [0, 1] \rightarrow \mathbb{R}^3$ be the path
  \[ \mathbf{x}(t) = (t, 3t^2, 2t^3) \]
- Then
  \[ \mathbf{x}'(t) = (1, 6t, 6t^2) \]

\[
\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{a}^{b} \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) dt \]
\[
= \int_{0}^{1} \left( t \mathbf{i} + 3t^2 \mathbf{j} + 2t^3 \mathbf{k} \right) \cdot \left( \mathbf{i} + 6t \mathbf{j} + 6t^2 \mathbf{k} \right) dt \\
= \int_{0}^{1} (t + 18t^3 + 12t^5) dt = \left( \frac{1}{2} t^2 + \frac{9}{2} t^4 + 2t^6 \right) \bigg|_{0}^{1} = 7
\]
Physical Interpretation of Vector Line Integrals

- Consider \( \mathbf{F} \) to be a force field in space.
- Then, the vector line integral could represent the work done by \( \mathbf{F} \) on a particle as the particle moves along the path \( \mathbf{x} \).

\[
\text{Total Work} = \int_{a}^{b} \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \, dt = \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}
\]

Simplified example

- Suppose \( \mathbf{F} \) is a constant vector field and \( \mathbf{x} \) is a straight-line.
Physical Interpretation of Vector Line Integrals

- Consider $\mathbf{F}$ to be a force field in space.
- Then, the vector line integral could represent the work done by $\mathbf{F}$ on a particle as the particle moves along the path $\mathbf{x}$.

\[
\text{Total Work} = \int_{a}^{b} \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) dt = \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}
\]

Simplified example

- Suppose $\mathbf{F}$ is a constant vector field and $\mathbf{x}$ is a straight-line.
- Then, the work done by $\mathbf{F}$ in moving a particle from one point $A$ along $\mathbf{x}$ to another point $B$ is given by

\[
\text{Work} = \mathbf{F} \cdot \Delta \mathbf{s} = \mathbf{F} \cdot (B - A)
\]
Differential Geometry Interpretation

- Suppose \( x : [a, b] \to \mathbb{R}^n \) is a \( C^1 \) path with \( x'(t) \neq 0 \) for \( a \leq t \leq b \).
- Recall that we define the **unit tangent vector** \( T \) to \( x \) by normalizing the velocity:
  \[
  T = \frac{x'(t)}{\|x'(t)\|}
  \]
- Then
  \[
  \int_x F \cdot ds = \int_a^b F(x(t)) \cdot x'(t) dt \\
  = \int_a^b (F(x(t)) \cdot T(t)) \|x'(t)\| dt = \int_x (F \cdot T) ds
  \]
Suppose \( \mathbf{x} : [a, b] \rightarrow \mathbb{R}^n \) is a \( C^1 \) path with \( x'(t) \neq 0 \) for \( a \leq t \leq b \). Then

\[
\int_{x} \mathbf{F} \cdot d\mathbf{s} = \int_{x} (\mathbf{F} \cdot \mathbf{T}) \, ds
\]

Since the dot product \( \mathbf{F} \cdot \mathbf{T} \) is a scalar quantity, we have written the original vector line integral as a scalar line integral.

It represents the (scalar) line integral of the tangential component of \( \mathbf{F} \) along the path.
Differential Geometry Interpretation

- Suppose \( \mathbf{x} : [a, b] \to \mathbb{R}^n \) is a \( C^1 \) path with \( \mathbf{x}'(t) \neq 0 \) for \( a \leq t \leq b \)

- Then

\[
\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} (\mathbf{F} \cdot \mathbf{T}) \, ds
\]
Example 4

- The circle \( x^2 + y^2 = 9 \) may be parametrized by

\[
\begin{align*}
  x &= 3 \cos t \\
  y &= 3 \sin t
\end{align*}
\]

, \( 0 \leq t \leq 2\pi \)

- Hence, a unit tangent vector is

\[
T = \frac{-3 \sin t \mathbf{i} + 3 \cos t \mathbf{j}}{\sqrt{9 \sin^2 t + 9 \cos^2 t}} = -\sin t \mathbf{i} + \cos t \mathbf{j} = \frac{-y \mathbf{i} + x \mathbf{j}}{3}
\]

- Now consider the radial vector field \( \mathbf{F} = x \mathbf{i} + y \mathbf{j} \) on \( \mathbb{R}^2 \)

- At every point along the circle we have

\[
\mathbf{F} \cdot T = (x \mathbf{i} + y \mathbf{j}) \cdot \left( \frac{-y \mathbf{i} + x \mathbf{j}}{3} \right) = 0
\]
Example 4

\[
\begin{align*}
\begin{cases}
x = 3 \cos t \\
y = 3 \sin t
\end{cases}, & 0 \leq t \leq 2\pi, \quad \mathbf{T} = -\frac{y}{3} \mathbf{i} + \frac{x}{3} \mathbf{j} \quad \text{and} \quad \mathbf{F} = x \mathbf{i} + y \mathbf{j} \Rightarrow \mathbf{F} \cdot \mathbf{T} = 0
\end{align*}
\]

Thus, \( \mathbf{F} \) is always perpendicular to the curve, and

\[
\int_{x} \mathbf{F} \cdot d\mathbf{s} = \int_{x} (\mathbf{F} \cdot \mathbf{T}) \, ds = \int_{x} 0 \, ds = 0
\]

Considering \( \mathbf{F} \) as a force, no work is done.
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Scalar and Vector Line Integrals

Green’s Theorem

Differential form of the line integral

Differential Form of the Line Integral

- Suppose that $\mathbf{x}(t) = (x(t), y(t), z(t)), a \leq t \leq b$, is a $C^1$ path
- Consider a continuous vector field $\mathbf{F}$ written as

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$$

- Then, from Definition 1.2 of the vector line integral, we have

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{a}^{b} (M(x, y, z)i + N(x, y, z)j + P(x, y, z)k) \cdot (x'(t)i + y'(t)j + z'(t)k) \, dt$$

$$= \int_{a}^{b} (M(x, y, z)x'(t) + N(x, y, z)y'(t) + P(x, y, z)z'(t)) \, dt$$

Recall that $dx = x'(t)dt, dy = y'(t)dt, dz = z'(t)dt$

$$= \int_{x} M(x, y, z)dx + N(x, y, z)dy + P(x, y, z)dz$$
Differential Form of the Line Integral

- Suppose that \( \mathbf{x}(t) = (x(t), y(t), z(t)), a \leq t \leq b \), is a \( C^1 \) path
- Consider a continuous vector field \( \mathbf{F} \) written as
  \[
  \mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}
  \]
- Then, from Definition 1.2 of the vector line integral, we have
  \[
  \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} M(x, y, z)dx + N(x, y, z)dy + P(x, y, z)dz
  \]
- A notational alternative is
  \[
  \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} M \, dx + N \, dy + P \, dz
  \]

The differential form of the line integral
Suppose that \( \mathbf{x}(t) = (x(t), y(t), z(t)), a \leq t \leq b \), is a \( C^1 \) path.

Consider a continuous vector field \( \mathbf{F} \) written as

\[
\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}
\]

Then, from Definition 1.2 of the vector line integral, we have

\[
\int \mathbf{F} \cdot d\mathbf{s} = \int M(x, y, z)\,dx + N(x, y, z)\,dy + P(x, y, z)\,dz
\]

A alternative notation is

\[
\int \mathbf{F} \cdot d\mathbf{s} = \int M\,dx + N\,dy + P\,dz
\]

\( M\,dx + N\,dy + P\,dz \) is itself called a differential form.

\( M\,dx + N\,dy + P\,dz \) should be evaluated using the parametric equations for \( x, y, \) and \( z \).
### Example 5

- Let \( \mathbf{x} \) be the path \( \mathbf{x}(t) = (t, t^2, t^3) \) for \( 0 \leq t \leq 1 \)
- We compute
  \[
  \int_{\mathbf{x}} (y + z) \, dx + (x + z) \, dy + (x + y) \, dz
  \]
- Along the path, we have
  \[
  x = t \Rightarrow dx = dt, y = t^2 \Rightarrow dy = 2t \, dt, z = t^3 \Rightarrow dz = 3t^2 \, dt
  \]
- Therefore
  \[
  \int_{\mathbf{x}} (y + z) \, dx + (x + z) \, dy + (x + y) \, dz
  = \int_0^1 (t^2 + t^3) \, dt + (t + t^3)2t \, dt + (t + t^2)3t^2 \, dt
  = \int_0^1 (5t^4 + 4t^3 + 3t^2) \, dt = (t^5 + t^4 + t^3)\Big|_0^1 = 3
  \]
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The Effect of Reparametrization

- The unit tangent vector to a path depends on the geometry of the underlying curve

  It doesn’t depend on the particular parametrization

- We might expect the line integral likewise to depend only on the image curve

- For example, consider the following two paths in the plane

  \[
  \begin{align*}
  x &: [0, 2\pi] \to \mathbb{R}^2, \quad x(t) = (\cos t, \sin t) \\
  y &: [0, \pi] \to \mathbb{R}^2, \quad y(t) = (\cos 2t, \sin 2t)
  \end{align*}
  \]

- Both \( x \) and \( y \) trace out a circle once in a counterclockwise sense

- If we let \( u(t) = 2t \), then we see that \( y(t) = x(u(t)) \)
Definition 1.3

- Let $\mathbf{x} : [a, b] \to \mathbb{R}^n$ be a piecewise $C^1$ path
- Consider another $C^1$ path $\mathbf{y} : [c, d] \to \mathbb{R}^n$
- We say that $\mathbf{y}$ is a reparametrization of $\mathbf{x}$ if there is a one-one and onto function $u : [c, d] \to [a, b]$ of class $C^1$
  - With inverse $u^{-1} : [a, b] \to [c, d]$ that is also of class $C^1$
  - Such that $\mathbf{y}(t) = \mathbf{x}(u(t))$, that is, $\mathbf{y} = \mathbf{x} \circ u$

Remark

- Thus, any reparametrization of a path must have the same underlying image curve as the original path
Example 6

- Consider the path

\[ \mathbf{x}(t) = (1 + 2t, 2 - t, 3 + 5t), \quad 0 \leq t \leq 1 \]

- It traces the line segment from the point (1, 2, 3) to the point (3, 1, 8)

1. So does the path

\[ \mathbf{y}(t) = (1 + 2t^2, 2 - t^2, 3 + 5t^2), \quad 0 \leq t \leq 1 \]

- We have that \( \mathbf{y} \) is a reparametrization of \( \mathbf{x} \) via the change of variable

\[ u(t) = t^2 \]
Example 6

- Consider the path

\[ x(t) = (1 + 2t, 2 - t, 3 + 5t), \quad 0 \leq t \leq 1 \]

- It traces the line segment from the point \((1, 2, 3)\) to the point \((3, 1, 8)\)

2. We consider now the path \(z : [-1, 1] \rightarrow \mathbb{R}^3\)

\[ z(t) = (1 + 2t^2, 2 - t^2, 3 + 5t^2), \quad -1 \leq t \leq 1 \]

- It is not a reparametrization of \(x\)
- We also have \(z(t) = x(u(t))\), where \(u(t) = t^2\)
- But in this case \(u\) maps \([-1, 1]\) onto \([0, 1]\) in a way that is not one-one
Example 6

- Consider the path

\[ x(t) = (1 + 2t, 2 - t, 3 + 5t), \quad 0 \leq t \leq 1 \]

- It traces the line segment from the point \((1, 2, 3)\) to the point \((3, 1, 8)\)

3. We finally consider the path \( w : [0, 1] \rightarrow \mathbb{R}^3 \)

\[ w(t) = (3 - 2t, 1 + t, 8 - 5t), \quad 0 \leq t \leq 1 \]

- It is a reparametrization of \( x \)
- We have \( w(t) = x(1 - t) \)
- So the function \( u : [0, 1] \rightarrow [0, 1] \) given by \( u(t) = 1 - t \) provides the change of variable for the reparametrization.

Geometrically, \( w \) traces the line segment between \((1, 2, 3)\) and \((3, 1, 8)\) in the opposite direction to \( x \)
Let \( y : [c, d] \to \mathbb{R}^n \) be a reparametrization of \( x : [a, b] \to \mathbb{R}^n \) via the change of variable \( u : [c, d] \to [a, b] \).

Then, since \( u \) is one-one, onto, and continuous, we must have either

1. \( u(c) = a \) and \( u(d) = b \), or
2. \( u(c) = b \) and \( u(d) = a \)

In case 1, we say that \( y \) (or \( u \)) is orientation-preserving.

\( y \) traces out the same image curve in the same direction that \( x \) does.

In case 2, we say that \( y \) (or \( u \)) is orientation-reversing.

\( y \) traces out the same image curve in the opposite direction that \( x \) does.
Example 7

- Let \( x : [a, b] \rightarrow \mathbb{R}^n \) be any \( C^1 \) path
- Then, we may define the opposite path \( x_{opp} : [a, b] \rightarrow \mathbb{R}^n \) by
  \[ x_{opp}(t) = x(a + b - t) \]

That is, \( x_{opp}(t) = x(u(t)) \), where \( u : [a, b] \rightarrow [a, b] \) is given by
  \[ u(t) = a + b - t \]

Clearly, then, \( x_{opp}(t) \) is an orientation-reversing reparametrization of \( x \).
Reparametrization and Velocity

In addition to reversing orientation, a reparametrization of a path can change the speed.

- This follows readily from the chain rule:
  \[
  \text{Speed of } y = \| y'(t) \| = \| u'(t) \| \| x'(t) \| = \| u'(t) \| \cdot (\text{Speed of } x)
  \]

- Since \( u \) is one-one, it follows that either:
  - \( u'(t) \geq 0 \) for all \( t \in [a, b] \) or
  - \( u'(t) \leq 0 \) for all \( t \in [a, b] \)

- The first case occurs when \( y \) is orientation-preserving
- The second case occurs when \( y \) is orientation-reversing.
Theorem 1.4

- Let \( x : [a, b] \to \mathbb{R}^n \) be a piecewise \( C^1 \) path
- Let \( f : X \subseteq \mathbb{R}^n \to \mathbb{R} \) be a continuous function whose domain \( X \) contains the image of \( x \)
- If \( y : [c, d] \to \mathbb{R}^n \) is any reparametrization of \( x \), then

\[
\int_y f \, ds = \int_x f \, ds
\]

Remark

- Theorems 1.4 tell us that scalar line integrals are independent of the way we might choose to reparametrize a path
Theorem 1.5

- Let \( x : [a, b] \rightarrow \mathbb{R}^n \) be a piecewise \( C^1 \) path.
- Let \( F : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a continuous vector field whose domain \( X \) contains the image of \( x \).
- If \( y : [c, d] \rightarrow \mathbb{R}^n \) is any reparametrization of \( x \), then
  1. If \( y \) is orientation-preserving, then
     \[
     \int_y F \cdot ds = \int_x F \cdot ds
     \]
  2. If \( y \) is orientation-reversing, then
     \[
     \int_y F \cdot ds = -\int_x F \cdot ds
     \]
Theorem 1.5

- Let \( x : [a, b] \rightarrow \mathbb{R}^n \) be a piecewise \( C^1 \) path.
- Let \( F : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a continuous vector field whose domain \( X \) contains the image of \( x \).
- If \( y : [c, d] \rightarrow \mathbb{R}^n \) is any reparametrization of \( x \), then

\[
\int_y F \cdot ds = \int_x F \cdot ds \quad \text{or} \quad \int_y F \cdot ds = -\int_x F \cdot ds
\]

Remark

- Theorems 1.5 tell us that vector line integrals are independent of reparametrization up to a sign.
- This sign depends only on whether the reparametrization preserves or reverses orientation.
Example 8

- Let \( \mathbf{F} = xi + yj \), and consider the following three paths between \((0, 0)\) and \((1, 1)\)

\[
\begin{align*}
x(t) &= (t, t), & 0 \leq t \leq 1 \\
y(t) &= (2t, 2t), & 0 \leq t \leq \frac{1}{2} \\
z(t) &= (1 - t, 1 - t), & 0 \leq t \leq 1
\end{align*}
\]

- The three paths are all reparametrizations of one another
- \( x, y, \) and \( z \) all trace the line segment between \((0, 0)\) and \((1, 1)\)
  - \( x \) and \( y \) from \((0, 0)\) to \((1, 1)\), and
  - \( z \) from \((1, 1)\) to \((0, 0)\)

- We can compare the values of the line integrals of \( \mathbf{F} \) along these paths
- The results of these calculations must agree with what Theorem 1.5 predicts
Example 8

Let \( \mathbf{F} = x\mathbf{i} + y\mathbf{j} \), and consider the following three paths between \((0, 0)\) and \((1, 1)\)

\[
\begin{align*}
\mathbf{x}(t) &= (t, t), & 0 \leq t \leq 1 \\
\mathbf{y}(t) &= (2t, 2t), & 0 \leq t \leq \frac{1}{2} \\
\mathbf{z}(t) &= (1 - t, 1 - t), & 0 \leq t \leq 1
\end{align*}
\]

\[
\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{0}^{1} \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\,dt = \int_{0}^{1} (t\mathbf{i} + t\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j})\,dt
\]

\[
= \int_{0}^{1} 2t \,dt = t^2\bigg|_{0}^{1} = 1
\]
Example 8

Let \( \mathbf{F} = xi + yj \), and consider the following three paths between \((0, 0)\) and \((1, 1)\)

\[
\begin{align*}
    \mathbf{x}(t) &= (t, t), & 0 \leq t \leq 1 \\
    \mathbf{y}(t) &= (2t, 2t), & 0 \leq t \leq \frac{1}{2} \\
    \mathbf{z}(t) &= (1 - t, 1 - t), & 0 \leq t \leq 1
\end{align*}
\]

\[
\int_{\gamma} \mathbf{F} \cdot d\mathbf{s} = \int_{0}^{\frac{1}{2}} \mathbf{F}(\mathbf{y}(t)) \cdot \mathbf{y}'(t) dt = \int_{0}^{\frac{1}{2}} (2t\mathbf{i} + 2t\mathbf{j}) \cdot (2\mathbf{i} + 2\mathbf{j}) dt \\
= \int_{0}^{\frac{1}{2}} 8t \ dt = 4t^2 \bigg|_{0}^{\frac{1}{2}} = 1
\]
Example 8

Let \( \mathbf{F} = x\mathbf{i} + y\mathbf{j} \), and consider the following three paths between \((0, 0)\) and \((1, 1)\)

\[
\begin{align*}
\mathbf{x}(t) &= (t, t), & 0 \leq t \leq 1 \\
\mathbf{y}(t) &= (2t, 2t), & 0 \leq t \leq \frac{1}{2} \\
\mathbf{z}(t) &= (1 - t, 1 - t), & 0 \leq t \leq 1
\end{align*}
\]

\[
\int_{\mathbf{z}} \mathbf{F} \cdot d\mathbf{s} = \int_{0}^{1} \mathbf{F}(\mathbf{z}(t)) \cdot \mathbf{z}'(t) dt
\]

\[
= \int_{0}^{1} ((1 - t)i + (1 - t)j) \cdot (-i - j) dt
\]

\[
= \int_{0}^{1} 2(t - 1) dt = (t - 1)^2\bigg|_{0}^{1} = -1
\]
Outline

1. Scalar and Vector Line Integrals
   - Scalar line integral
   - Vector line integral
   - Differential form of the line integral
   - Effect of reparametrization
   - Closed and simples curves

2. Green’s Theorem
   - Definition
   - Examples
Theorems 1.4 and 1.5 enable us to define line integrals over curves rather than over parametrized paths.

To be more explicit, we say that a piecewise $C^1$ path $\mathbf{x} : [a, b] \to \mathbb{R}^n$ is closed if $\mathbf{x}(a) = \mathbf{x}(b)$.

We say that the path $\mathbf{x}$ is simple if it has no self-intersections.

That is, if $\mathbf{x}$ is one-one on $[a, b]$, except possibly that $\mathbf{x}(a)$ may equal $\mathbf{x}(b)$.

Then, by a curve $C$, we now mean the image of a path $\mathbf{x} : [a, b] \to \mathbb{R}^n$.

This path is one-one except possibly at finitely many points of $[a, b]$.

The (nearly) one-one path $\mathbf{x}$ will be called a parametrization of $C$.
Closed and Simple Curves

- Not simple, not closed
- Simple, not closed
- Not simple, closed
- Simple, closed
Example 9

Consider the ellipse

\[ \frac{x^2}{25} + \frac{y^2}{9} = 1 \]

It is a simple, closed curve that may be parametrized by either

\[
x(t) = (5 \cos t, 3 \sin t), \quad x : [0, 2\pi] \to \mathbb{R}^2
\]

or

\[
y(t) = (5 \cos 2(\pi - t), 3 \sin 2(\pi - t)), \quad y : [0, \pi] \to \mathbb{R}^2
\]
Example 9

- Consider the ellipse

\[ \frac{x^2}{25} + \frac{y^2}{9} = 1 \]

- Consider now the path

\[ z(t) = (5 \cos t, 3 \sin t), \quad z : [0, 6\pi] \rightarrow \mathbb{R}^2 \]

- It is not a parametrization, since it traces the ellipse three times as \( t \) increases from 0 to \( 6\pi \). \( z \) is not one-one.
Example 10

- Let $C$ be the upper semicircle of radius 2, centered at $(0, 0)$ and oriented counterclockwise from $(2, 0)$ to $(-2, 0)$.
- We calculate
  \[ \int_C (x^2 - y^2 + 1) \, ds \]
- We can choose any parametrization for $C$, for instance,
  \[
  \begin{align*}
  \mathbf{x}(t) & = (2 \cos t, 2 \sin t), \quad 0 \leq t \leq \pi \\
  \text{or} \\
  \mathbf{y}(t) & = (-2 \cos 2t, -2 \sin 2t), \quad -\frac{\pi}{2} \leq t \leq 0
  \end{align*}
  \]
- Note that $\mathbf{y}(t) = \mathbf{x}(2t + \pi)$.
Example 10

Let $C$ be the upper semicircle of radius 2, centered at $(0, 0)$ and oriented counterclockwise from $(2, 0)$ to $(-2, 0)$.

We calculate

$$\int_C (x^2 - y^2 + 1) \, ds$$

$$x(t) = (2 \cos t, 2 \sin t), \quad 0 \leq t \leq \pi$$

Then

$$\int_C (x^2 - y^2 + 1) \, ds = \int_x (x^2 - y^2 + 1) \, ds$$

$$= \int_0^\pi (4 \cos^2 t - 4 \sin^2 t + 1) \sqrt{4 \sin^2 t + 4 \cos^2 t} \, dt$$

By the double-angle formula $\cos(2t) = \cos^2 t - \sin^2 t$

$$= \int_0^\pi (4 \cos 2t + 1) 2 \, dt = 2 (\sin 2t + t)|_0^\pi = 2\pi$$
Example 10

- Let $C$ be the upper semicircle of radius 2, centered at $(0, 0)$ and oriented counterclockwise from $(2, 0)$ to $(-2, 0)$
- We calculate
  \[
  \int_C (x^2 - y^2 + 1) \, ds
  \]
- $y(t) = (-2 \cos 2t, -2 \sin 2t), \quad -\frac{\pi}{2} \leq t \leq 0$
- Then
  \[
  \int_C (x^2 - y^2 + 1) \, ds = \int_y (x^2 - y^2 + 1) \, ds
  \]
  \[
  = \int_{-\pi/2}^{0} (4 \cos^2 2t - 4 \sin^2 2t + 1) \sqrt{16 \sin^2 2t + 16 \cos^2 2t} \, dt
  \]
  By the double-angle formula
  \[
  = \int_{-\pi/2}^{0} (4 \cos 4t + 1) \, 4 \, dt = 4 (\sin 4t + t)|_{-\pi/2}^{0} = 2\pi
  \]
Example 11

Consider the force

\[ \mathbf{F} = x\mathbf{i} - y\mathbf{j} + (x + y + z)\mathbf{k} \]

We calculate the work done by the force \( \mathbf{F} \) on a particle that moves

- Along the parabola \( y = 3x^2 \), \( z = 0 \)
- From the origin to the point \( (2, 12, 0) \)
Example 11

Consider the force

\[ \mathbf{F} = x \mathbf{i} - y \mathbf{j} + (x + y + z) \mathbf{k} \]

Along \( y = 3x^2, \ z = 0 \), from \((0, 0, 0)\) to \((2, 12, 0)\)

We parametrize the parabola by

\[ x = t, \ y = 3t^2, \ z = 0 \] for \( 0 \leq t \leq 2 \)

Then, by Definition 1.2

\[
\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^2 \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \, dt
\]

\[
= \int_0^2 \left( t, -3t^2, t + 3t^2 \right) \cdot (1, 6t, 0) \, dt = \int_0^2 (t - 18t^3) \, dt
\]

\[
= \left( \frac{1}{2} t^2 - \frac{9}{2} t^4 \right) \bigg|_0^2 = 2 - 72 = -70
\]
Example 11

Consider the force

\[ \mathbf{F} = x \mathbf{i} - y \mathbf{j} + (x + y + z) \mathbf{k} \]

Along \( y = 3x^2, \ z = 0 \), from \((0, 0, 0)\) to \((2, 12, 0)\)

We parametrize the parabola by

\[ x = t, \ y = 3t^2, \ z = 0 \text{ for } 0 \leq t \leq 2 \]

Then, by Definition 1.2

\[
\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{s} = \int_x \mathbf{F} \cdot d\mathbf{s} = \int_0^2 \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \, dt = -70
\]

The meaning of the negative sign is that by moving along the curve in the indicated direction, work is done against the force.
Example 11

- Consider the force
  \[ \mathbf{F} = x \mathbf{i} - y \mathbf{j} + (x + y + z) \mathbf{k} \]
  Along \( y = 3x^2, \, z = 0 \), from \((0, 0, 0)\) to \((2, 12, 0)\)

- We parametrize the parabola by \( x = t, \, y = 3t^2, \, z = 0 \) for \( 0 \leq t \leq 2 \)

- Then, by Definition 1.2
  \[
  \text{Work} = \int_C \mathbf{F} \cdot d\mathbf{s} = \int_x \mathbf{F} \cdot d\mathbf{s} = \int_0^2 \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \, dt = -70
  \]

- If we orient the curve the opposite way, then the work done in moving from \((2, 12, 0)\) to \((0, 0, 0)\) would be 70
Outline

1. Scalar and Vector Line Integrals
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   - Closed and simples curves

2. Green’s Theorem
   - Definition
   - Examples
Theorem 2.1: Green’s Theorem

- Let $D$ be a closed, bounded region in $\mathbb{R}^2$
- Assume its boundary $C = \partial D$ consists of finitely many simple, closed, piecewise $C^1$ curves
- Orient the curves of $C$ so that $D$ is on the left as one traverses $C$

If $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is a vector field of class $C^1$ throughout $D$, then

$$\oint_C M\,dx + N\,dy = \iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \,dxdy$$
Theorem 2.1: Green’s Theorem

If \( \mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j} \) is a vector field of class \( C^1 \) throughout \( D \), then

\[
\int_C M\,dx + N\,dy = \int \int_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \,dxdy
\]

The symbol \( \oint_C \) indicates that the line integral is taken over one or more closed curves.

Green’s Theorem relates the vector line integral around a closed curve \( C \) in \( \mathbb{R}^2 \) to an appropriate double integral over the plane region \( D \) bounded by \( C \).
Outline

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2. Green’s Theorem
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Example 1

Let \( \mathbf{F} = xy \mathbf{i} + y^2 \mathbf{j} \) and let \( D \) be the first quadrant region bounded by the line \( y = x \) and the parabola \( y = x^2 \).

\( \partial D \) is oriented counterclockwise, the orientation stipulated by the statement of Green’s Theorem.

We can calculate

\[
\oint_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \int_{\partial D} xy \, dx + y^2 \, dy
\]
Example 1

Let \( \mathbf{F} = xy \mathbf{i} + y^2 \mathbf{j} \) and let \( D \) be the first quadrant region bounded by the line \( y = x \) and the parabola \( y = x^2 \).

We need to parametrize the two \( C^1 \) pieces of \( \partial D \) separately.

\[ C_1 : \begin{cases} x = t \\ y = t^2 \end{cases} , \quad 0 \leq t \leq 1 \quad \text{and} \quad C_2 : \begin{cases} x = 1 - t \\ y = 1 - t \end{cases} , \quad 0 \leq t \leq 1 \]

Note the orientations of \( C_1 \) and \( C_2 \).
Example 1

\( \mathbf{F} = xy \mathbf{i} + y^2 \mathbf{j} \),  \( D \) be the first quadrant bounded by \( y = x \) and \( y = x^2 \)

\( C_1 : \begin{cases} x = t \\ y = t^2 \end{cases} \), \( 0 \leq t \leq 1 \) and \( C_2 : \begin{cases} x = 1 - t \\ y = 1 - t \end{cases} \), \( 0 \leq t \leq 1 \)

Then

\[
\oint_{\partial D} xy \, dx + y^2 \, dy = \oint_{C_1} xy \, dx + y^2 \, dy + \oint_{C_2} xy \, dx + y^2 \, dy
\]

\[
= \int_0^1 (t \cdot t^2 + t^4 \cdot 2t) \, dt + \int_0^1 ((1 - t)^2 + (1 - t)^2) \, (-dt)
\]

\[
= \int_0^1 (t^3 + 2t^5) \, dt + \int_0^1 2(1 - t)^2 \, (-dt)
\]

\[
= \left( \frac{1}{4} t^4 + \frac{2}{6} t^6 \right) \bigg|_0^1 + \left( \frac{2}{3} (1 - t)^3 \right) \bigg|_0^1 = \frac{1}{4} + \frac{2}{6} - \frac{2}{3} = -\frac{1}{12}
\]
Example 1

\[ \mathbf{F} = xy \mathbf{i} + y^2 \mathbf{j}, \quad D \text{ be the first quadrant bounded by } y = x \text{ and } y = x^2 \]

\[ C_1 : \begin{cases} 
  x = t \\
  y = t^2 
\end{cases}, \quad 0 \leq t \leq 1 \quad \text{and} \quad C_2 : \begin{cases} 
  x = 1 - t \\
  y = 1 - t 
\end{cases}, \quad 0 \leq t \leq 1 \]

On the other hand

\[
\int \int_D \left( \frac{\partial}{\partial x} (y^2) - \frac{\partial}{\partial y} (xy) \right) \, dx \, dy = \int_0^1 \int_{x^2}^x -x \, dy \, dx
\]

\[
= \int_0^1 -x (x - x^2) \, dx = \int_0^1 (x^3 - x^2) \, dx = \left( \frac{1}{4}x^4 - \frac{1}{3}x^3 \right) \bigg|_0^1
\]

\[
= \frac{1}{4} - \frac{1}{3} = -\frac{1}{12}
\]

The line integral and the double integral agree.
Example 2

- Let $C$ be the circle of radius $a$, oriented counterclockwise.
- Then, $C$ is the boundary of the disk $D$ of radius $a$.

We calculate the line integral

$$\oint_C -y \, dx + x \, dy$$

- Although we can parametrize $C$ and thus evaluate the line integral, it is easier to employ Green’s Theorem instead.
Example 2

- Let $C$ be the circle of radius $a$, oriented counterclockwise.
- Then, $C$ is the boundary of the disk $D$ of radius $a$.

We calculate line integral

\[ \oint_C -y\, dx + x\, dy = \iint_D \left( \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) \right)\, dx\, dy \]

\[ = \iint_D 2\, dx\, dy = 2(\text{Area of } D) = 2\pi a^2 \]
Generalization of Example 2

- Suppose $D$ is any region to which Green’s Theorem can be applied.
- Then, orienting $\partial D$ appropriately, we have

$$\frac{1}{2} \oint_{\partial D} - y \, dx + x \, dy = \frac{1}{2} \int \int_{D} 2 \, dx \, dy = \text{Area of } D$$

- Thus, we can calculate the area of a region (two-dimensional) by using line integrals (one-dimensional)
Example 3

We compute the area inside the ellipse

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

The ellipse itself may be parametrized counterclockwise by

\[
\begin{cases} 
  x = a \cos t \\
  y = b \sin t 
\end{cases}, \quad 0 \leq t \leq 2\pi
\]
Example 3

- We compute the area inside the ellipse
  \[
  \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \begin{cases} x = a \cos t \\ y = b \sin t \end{cases}, \quad 0 \leq t \leq 2\pi
  \]

- Then
  \[
  \text{Area of ellipse} = \frac{1}{2} \oint_{\partial D} -y \, dx + x \, dy
  \]
  \[
  = \frac{1}{2} \int_0^{2\pi} -b \sin t (-a \sin t \, dt) + a \cos t (b \cos t \, dt)
  \]
  \[
  = \frac{1}{2} \int_0^{2\pi} (ab \sin^2 t + ab \cos^2 t) \, dt = \frac{1}{2} \int_0^{2\pi} ab \, dt = \pi ab
  \]