Métodos Matemáticos de Bioingeniería

Grado en Ingeniería Biomédica
Lecture 22

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Outline

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Path-Independent Line Integrals

- Line integrals of a given vector field depend only on the underlying curve and its orientation

  Not on the particular parametrization of the curve

- In some special instances, however, even the curve itself doesn't matter

  Only the initial and terminal points are relevant

- A vector field is said to have path-independent line integrals if it has the property that

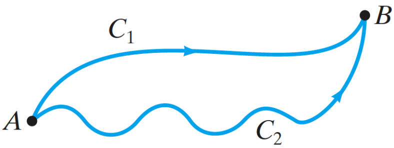
  Line integrals of it depend only on the initial and terminal points of the oriented curve over which the line integral is taken

Definition 3.1

- A continuous vector field $\mathbf{F}$ has path-independent line integrals if

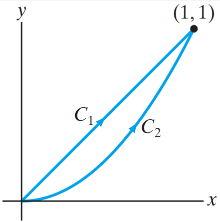
$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$$

for any two simple, piecewise C^1, oriented curves lying in the domain of $\mathbf{F}$ and having the same initial and terminal points



Example 1

- Let $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ and consider the following two curves in $\mathbb{R}^2$ from the origin to $(1, 1)$:
  - C_1, the line segment f rom $(0, 0)$ to $(1, 1)$, and
  - C_2, the portion of the parabola $y = x^2$



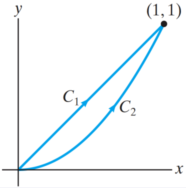
- These curves may be parametrized as

$$C_1 : \begin{cases} x = t \\ y = t \end{cases}, 0 \leq t \leq 1 \text{ and } C_2 : \begin{cases} x = t \\ y = t^2 \end{cases}, 0 \leq t \leq 1$$

Example 1

$$\mathbf{F} = y\mathbf{i} - x\mathbf{j}$$

$$C_1 : \begin{cases} x = t \\ y = t \end{cases}, 0 \leq t \leq 1 \text{ and } C_2 : \begin{cases} x = t \\ y = t^2 \end{cases}, 0 \leq t \leq 1$$



- Then, we calculate

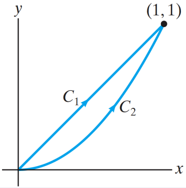
$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (t\mathbf{i} - t\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j}) dt = \int_0^1 0 \; dt = 0$$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (t^2\mathbf{i} - t\mathbf{j}) \cdot (\mathbf{i} + 2t\mathbf{j}) dt = \int_0^1 \left(t^2 - 2t^2\right) dt = -\frac{1}{3} t^3 \Big|_0^1 = -\frac{1}{3}$$

Example 1

$$\mathbf{F} = y\mathbf{i} - x\mathbf{j}$$

$$C_1 : \begin{cases} x = t \\ y = t \end{cases}, 0 \leq t \leq 1 \quad \text{and} \quad C_2 : \begin{cases} x = t \\ y = t^2 \end{cases}, 0 \leq t \leq 1$$



- Thus

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$$

- And so **F** does not have path-independent line integrals

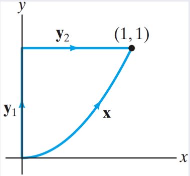
Example 2

- It can be shown that vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ has path-independent line integrals
- We illustrate this fact by considering
  - The parabolic path $\mathbf{x} : [0,1] \to \mathbb{R}^2, \quad \mathbf{x}(t) = (t, t^2)$
  - The path $\mathbf{y} : [0,2] \to \mathbb{R}^2$ made up of the two straight segments

$$\mathbf{y}_1 : [0,1] \to \mathbb{R}^2, \quad \mathbf{y}_1(t) = (0, t)$$

and

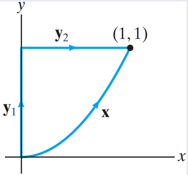
$$\mathbf{y}_2 : [1,2] \to \mathbb{R}^2, \quad \mathbf{y}_2(t) = (t-1, 1)$$



- Both $\mathbf{x}$ and $\mathbf{y}$ are paths from $(0,0)$ to $(1,1)$

Example 2

$$\mathbf{F} = x\mathbf{i} + y\mathbf{j}$$
$$\mathbf{x}(t) = (t, t^2)$$
$$\mathbf{y}_1 : [0,1] \to \mathbb{R}^2, \quad \mathbf{y}_1(t) = (0, t)$$
$$\mathbf{y}_2 : [1,2] \to \mathbb{R}^2, \quad \mathbf{y}_2(t) = (t-1, 1)$$

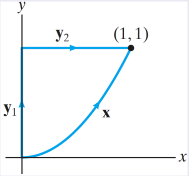


- Then

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (t\mathbf{i} + t^2\mathbf{j}) \cdot (\mathbf{i} + 2t\mathbf{j})dt = \int_0^1 \left(t + 2t^3\right) dt = \left. \frac{1}{2}t^2 + \frac{1}{2}t^4 \right|_0^1 = 1$$

Example 2

$$\mathbf{F} = x\mathbf{i} + y\mathbf{j}$$
$$\mathbf{x}(t) = (t, t^2)$$
$$\mathbf{y}_1 : [0,1] \to \mathbb{R}^2, \quad \mathbf{y}_1(t) = (0, t)$$
$$\mathbf{y}_2 : [1,2] \to \mathbb{R}^2, \quad \mathbf{y}_2(t) = (t-1, 1)$$

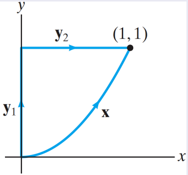


- Then

$$\int_{\mathbf{y}} \mathbf{F}\cdot d\mathbf{s} = \int_{\mathbf{y}_1} \mathbf{F}\cdot d\mathbf{s} + \int_{\mathbf{y}_2} \mathbf{F}\cdot d\mathbf{s} = \int_0^1 t\mathbf{j}\cdot\mathbf{j}\, dt + \int_1^2 ((t-1)\mathbf{i} + \mathbf{j})\cdot\mathbf{i}\, dt$$
$$= \int_0^1 t\, dt + \int_1^2 (t-1)\, dt = \frac{1}{2}t^2\Big|_0^1 + \frac{1}{2}(t-1)^2\Big|_1^2 = \frac{1}{2} + \frac{1}{2} = 1$$

Example 2

$$\mathbf{F} = x\mathbf{i} + y\mathbf{j}$$
$$\mathbf{x}(t) = (t, t^2)$$
$$\mathbf{y}_1 : [0,1] \to \mathbb{R}^2, \quad \mathbf{y}_1(t) = (0, t)$$
$$\mathbf{y}_2 : [1,2] \to \mathbb{R}^2, \quad \mathbf{y}_2(t) = (t-1, 1)$$



- Thus

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{y}} \mathbf{F} \cdot d\mathbf{s}$$

Checking that the value of the line integral of **F** along any choice of path between any two points is the same as any other, is a prohibitive task

Theorem 3.2

- Let $\mathbf{F}$ be a continuous vector field
- Then $\mathbf{F}$ has path-independent line integrals if and only if

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = 0$$

for all piecewise C^1, simple, closed curves C in the domain of the vector field $\mathbf{F}$

Remark

- This result is a reformulation of the path-independence property
- It is not essential to assume that the curves in Definition 3.1 and Theorem 3.2 are simple

Gradient Fields and Conservative Vector Fields

- Suppose that $\mathbf{F}$ is a continuous vector field such that $\mathbf{F} = \nabla f$, where f is some scalar-valued function of class C^1
- We call $\mathbf{F}$ a conservative vector field as well as a gradient field
- Recall that we refer to f as a scalar potential of $\mathbf{F}$
- Then, along any path $\mathbf{F}$ from $A = \mathbf{x}(a)$ to $B = \mathbf{x}(b)$ whose image lies in the domain of F

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} \nabla f \cdot d\mathbf{s} = \int_a^b \nabla f(\mathbf{x}(t)) \cdot \mathbf{x}'(t) dt$$

From the chain rule $d/dt\,[f(\mathbf{x}(t))] = \nabla f(\mathbf{x}(t)) \cdot \mathbf{x}'(t)$

$$= \int_a^b \frac{d}{dt}\,[f(\mathbf{x}(t))]\,dt = f(\mathbf{x}(t))\big|_a^b = f(\mathbf{x}(b)) - f(\mathbf{x}(a)) = f(B) - f(A)$$

Gradient Fields and Conservative Vector Fields

- Suppose that $\mathbf{F}$ is a continuous vector field such that $\mathbf{F} = \nabla f$, where f is some scalar-valued function of class C^1
- We call $\mathbf{F}$ a conservative vector field as well as a gradient field
- Recall that we refer to f as a scalar potential of $\mathbf{F}$
- Then, along any path $\mathbf{F}$ from $A = \mathbf{x}(a)$ to $B = \mathbf{x}(b)$ whose image lies in the domain of F

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{x}(b)) - f(\mathbf{x}(a)) = f(B) - f(A)$$

- The line integral of a gradient field $\mathbf{F}$ depends only on the value of the potential function at the endpoints of the path
- Hence, gradient fields have path-independent line integrals

The converse
holds as well

Theorem 3.3

- Let $\mathbf{F}$ be defined and continuous on a connected, open region R of $\mathbb{R}^n$
- Then $\mathbf{F} = \nabla f$ if and only if $\mathbf{F}$ has path-independent line integrals over curves in R
- Moreover, if C is any piecewise C^1, oriented curve lying in R with initial point A and terminal point B, then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = f(B) - f(A)$$

Remarks

- f must be a function of class C^1 on R
- A region $R \subseteq \mathbb{R}^n$ is connected if any two points in R can be joined by a path whose image lies in R

Example 3

- Consider the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ of Example 2 again
- It is easy to check that $\mathbf{F} = \nabla f$ where

$$f(x, y) = \frac{1}{2}\left(x^2 + y^2\right)$$

- By Theorem 3.3, line integrals of $\mathbf{F}$ will be path independent
- This fact was illustrated, but not proved, in Example 2
- Now, by Theorem 3.3, we see that for any directed piecewise C^1 curve C from $(0, 0)$ to $(1, 1)$, we have

$$\int_C \mathbf{F} \cdot d\mathbf{s} = f(1,1) - f(0,0) = \frac{1}{2}(1^2 + 1^2) - \frac{1}{2}(0^2 + 0^2) = 1$$

- This result agrees with our earlier computations

Outline

Two questions

- Theorem 3.3 tells us that
  - A vector field **F** has path-independent line integrals when it is a conservative (gradient) vector field
  - The line integral of **F** along any path is determined by the values of the potential function f at the endpoints of the path
- Two questions arise naturally:
  1. How can we determine whether a given vector field **F** is conservative?
  2. Assuming that **F** is conservative, is there a procedure for finding a scalar potential function f such that $\mathbf{F} = \nabla f$?
- We answer the first question by providing a simple and effective test that can be performed on **F**
- Should **F** pass this test then we illustrate via examples how to produce a scalar potential for **F**

Definition 3.4

- A region R in $\mathbb{R}^2$ or $\mathbb{R}^3$ is simply-connected if
  - It consists of a single connected piece, and
  - Every simple, closed curve C in R can be continuously shrunk to a point while remaining in R throughout the deformation
- In other words, R is simply-connected if
  - It is connected, and
  - Every simple, closed curve C lying in R has the property that all the points enclosed by C also lie in R

Loosely speaking, a simply-connected region can have no "essential holes"

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- A region R on $\mathbb{R}^2$ is simply-connected if
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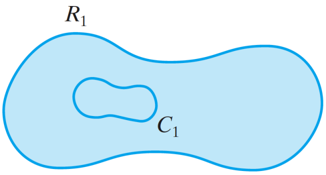

Simply-connected region

Definition 3.4

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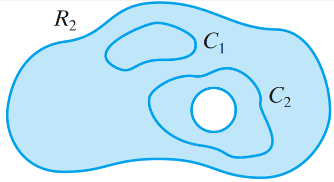

Not simply-connected region

Definition 3.4

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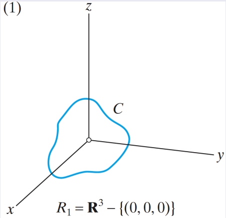

Simply-connected region

Definition 3.4

- A region R in $\mathbb{R}^2$ or $\mathbb{R}^3$ is simply-connected if
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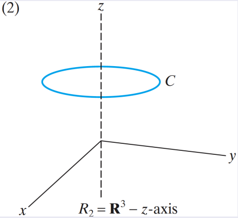

Not simply-connected region:
The curve C cannot be shrunk continuously to a point without becoming "stuck" on the "missing" z-axis.

Theorem 3.5

- Let $\mathbf{F}$ be a vector field of class C^1 whose domain is a simply-connected region R in either $\mathbb{R}^2$ or $\mathbb{R}^3$
- Then $\mathbf{F} = \nabla f$ for some scalar-valued function f of class C^2 on R if and only if $\nabla \times \mathbf{F} = \mathbf{0}$ at all points of R

Remarks

- Theorem 3.5 provides a straightforward way to determine if a vector field $\mathbf{F}$ is conservative:
  - Check that the domain of $\mathbf{F}$ is simply-connected
  - Test if $\nabla \times \mathbf{F} = \mathbf{0}$
  - If the curl vanishes, it follows that $\mathbf{F}$ has path-independent line integrals

Theorem 3.5

- Let $\mathbf{F}$ be a vector field of class C^1 whose domain is a simply-connected region R in either $\mathbb{R}^2$ or $\mathbb{R}^3$
- Then $\mathbf{F} = \nabla f$ for some scalar-valued function f of class C^2 on R if and only if $\nabla \times \mathbf{F} = \mathbf{0}$ at all points of R

Remarks

- Consider a two-dimensional vector field

$$\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

- The condition that the curl of $\mathbf{F}$ vanishes means

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} = \mathbf{0} \iff \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Example 4

- Let $\mathbf{F} = x^2y\mathbf{i} - 2xy\mathbf{j}$
- Then

$$\begin{aligned} \frac{\partial}{\partial x}(-2xy) &= -2y \\ \frac{\partial}{\partial y}(x^2y) &= x^2 \end{aligned}$$

- Thus

$$\frac{\partial}{\partial x}(-2xy) \neq \frac{\partial}{\partial y}(x^2y)$$

- Since these partial derivatives are not equal, we conclude that $\mathbf{F}$ is not conservative, by Theorem 3.5

Example 5

- Let $\mathbf{F} = (2xy + \cos 2y)\mathbf{i} + (x^2 - 2x\sin 2y)\mathbf{j}$
- This vector field $\mathbf{F}$ is defined and of class C^1 on all of $\mathbb{R}^2$
- $\mathbb{R}^2$ is a simply-connected region, and

$$\frac{\partial}{\partial x}(x^2 - 2x\sin 2y) = 2x - 2\sin 2y = \frac{\partial}{\partial y}(2xy + \cos 2y)$$

- We may conclude that $\mathbf{F}$ is conservative
- In addition, suppose C is the ellipse $x^2/4 + y^2 = 1$
- C is a simple, closed curve
- Then by Theorems 3.2 and 3.3, we conclude, without any explicit calculation, that

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = 0$$

Example 7

- Consider the vector field of Example 5

$$\mathbf{F} = (2xy + \cos 2y)\mathbf{i} + (x^2 - 2x\sin 2y)\mathbf{j}$$

- We have already seen that **F** is conservative
- To find a scalar potential for F, we seek a suitable function $f(x, y)$ such that

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = \mathbf{F}$$

- Components of the gradient of f must agree with those of **F**

$$\begin{cases} \frac{\partial f}{\partial x} = 2xy + \cos 2y \\ \frac{\partial f}{\partial y} = x^2 - 2x\sin 2y \end{cases}$$

Example 7

$$\begin{cases} \frac{\partial f}{\partial x} = 2xy + \cos 2y \\ \frac{\partial f}{\partial y} = x^2 - 2x \sin 2y \end{cases}$$

- We may begin to recover f by integrating the first equation with respect to x

$$f(x, y) = \int \frac{\partial f}{\partial x} dx = \int (2xy + \cos 2y) dx = x^2 y + x \cos 2y + g(y)$$

- $g(y)$ is an arbitrary function of y

The function $g(y)$ plays the role of the arbitrary "constant of integration" in the indefinite integral

- Differentiating this equation with respect to y yields

$$\frac{\partial f}{\partial y} = x^2 - 2x \sin 2y + g'(y)$$

Example 7

$$\begin{cases} \frac{\partial f}{\partial x} = 2xy + \cos 2y \\ \frac{\partial f}{\partial y} = x^2 - 2x \sin 2y \end{cases} \quad , \quad \frac{\partial f}{\partial y} = x^2 - 2x \sin 2y + g'(y)$$

- If we compare both equations implying $\frac{\partial f}{\partial y}$, we see that

$$g'(y) \equiv 0$$

- So g must be a constant function
- Therefore, the scalar potential must be of the form

$$f(x, y) = x^2 y + x \cos 2y + C$$

  where C is an arbitrary constant
- As a double-check, we can verify that $\nabla f = \mathbf{F}$

Example 8

- Let $\mathbf{F} = (e^x \sin y - yz)\mathbf{i} + (e^x \cos y - xz)\mathbf{j} + (z - xy)\mathbf{k}$
- Note that $\mathbf{F}$ is of class C^1 on all of $\mathbb{R}^3$
- We calculate

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y - yz & e^x \cos y - xz & z - xy \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(z - xy) - \frac{\partial}{\partial z}(e^x \cos y - xz) \right) \mathbf{i} \\
&+ \left(\frac{\partial}{\partial z}(e^x \sin y - yz) - \frac{\partial}{\partial x}(z - xy) \right) \mathbf{j} \\
&+ \left(\frac{\partial}{\partial x}(e^x \cos y - xz) - \frac{\partial}{\partial y}(e^x \sin y - yz) \right) \mathbf{k} = \mathbf{0}
\end{aligned}$$

- Therefore, by Theorem 3.5, $\mathbf{F}$ is conservative

Example 8

- Let $\mathbf{F} = (e^x \sin y - yz)\mathbf{i} + (e^x \cos y - xz)\mathbf{j} + (z - xy)\mathbf{k}$
- Any scalar potential $f(x, y, z)$ for $\mathbf{F}$ must satisfy

$$\begin{cases} \frac{\partial f}{\partial x} = e^x \sin y - yz \\ \frac{\partial f}{\partial y} = e^x \cos y - xz \\ \frac{\partial f}{\partial z} = z - xy \end{cases}$$

- Integrating $\partial f / \partial x$ with respect to x, we find that

$$f(x, y, z) = \int \frac{\partial f}{\partial x} dx = \int (e^x \sin y - yz) dx = e^x \sin y - xyz + g(y, z)$$

where $g(y, z)$ may be any function of y and z

Example 8

- Let $\mathbf{F} = (e^x \sin y - yz)\mathbf{i} + (e^x \cos y - xz)\mathbf{j} + (z - xy)\mathbf{k}$
- Any scalar potential $f(x, y, z)$ for $\mathbf{F}$ must satisfy

$$\begin{cases} \frac{\partial f}{\partial x} = e^x \sin y - yz \\ \frac{\partial f}{\partial y} = e^x \cos y - xz \\ \frac{\partial f}{\partial z} = z - xy \end{cases}, \quad f(x, y, z) = e^x \sin y - xyz + g(y, z)$$

- Differentiating equation now with respect to y and comparing with $\partial f/\partial y$

$$\frac{\partial f}{\partial y} = e^x \cos y - xz + \frac{\partial g}{\partial y} = e^x \cos y - xz \Rightarrow \frac{\partial g}{\partial y} = 0$$

- So g must be independent of y, that is, $g(y, z) = h(z)$, and

$$f(x, y, z) = e^x \sin y - xyz + h(z)$$

Example 8

- Let $\mathbf{F} = (e^x \sin y - yz)\mathbf{i} + (e^x \cos y - xz)\mathbf{j} + (z - xy)\mathbf{k}$
- Any scalar potential $f(x, y, z)$ for $\mathbf{F}$ must satisfy

$$\begin{cases} \frac{\partial f}{\partial x} = e^x \sin y - yz \\ \frac{\partial f}{\partial y} = e^x \cos y - xz \\ \frac{\partial f}{\partial z} = z - xy \end{cases}, \quad f(x, y, z) = e^x \sin y - xyz + h(z)$$

- Finally, we differentiate the equation with respect to z and compare with $\partial f / \partial z$

$$\frac{\partial f}{\partial z} = -xy + h'(z) = z - xy \Rightarrow h'(z) = z$$

- So

$$h(z) = \frac{1}{2} z^2 + C$$

where C is an arbitrary constant

Example 8

- Let $\mathbf{F} = (e^x \sin y - yz)\mathbf{i} + (e^x \cos y - xz)\mathbf{j} + (z - xy)\mathbf{k}$
- Any scalar potential $f(x, y, z)$ for $\mathbf{F}$ must satisfy

$$\begin{cases} \frac{\partial f}{\partial x} = e^x \sin y - yz \\ \frac{\partial f}{\partial y} = e^x \cos y - xz \\ \frac{\partial f}{\partial z} = z - xy \end{cases} , \quad f(x, y, z) = e^x \sin y - xyz + h(z)$$

$$h(z) = \frac{1}{2} z^2 + C$$

- Thus, a scalar potential for the original vector field $\mathbf{F}$ is given by

$$f(x, y, z) = e^x \sin y - xyz + \frac{1}{2} z^2 + C$$