Métodos Matemáticos de Bioingeniería

Grado en Ingeniería Biomédica

Lecture 3

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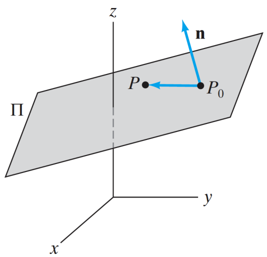
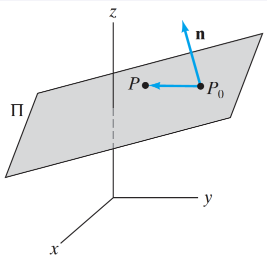
Planes in $\mathbb{R}^3$

- A plane Π in $\mathbb{R}^3$ is determined uniquely by the following geometric information:
  - A particular point $P_0(x_0, y_0, z_0)$ in the plane.
  - A particular vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ that is normal (perpendicular) to the plane.



Π is the set of all points $P(x, y, z)$ in space such that $\overrightarrow{P_0P}$ is perpendicular to $\mathbf{n}$

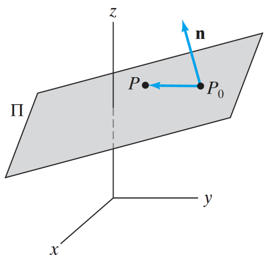
Planes in $\mathbb{R}^3$



- Π is defined by the vector equation

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0$$

Planes in $\mathbb{R}^3$



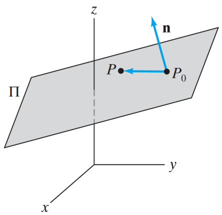
- Since $\overrightarrow{P_0P} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$ equation may be rewritten

$$(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot ((x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}) = 0$$

or

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Planes in $\mathbb{R}^3$



$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

- This is equivalent to

$$Ax + By + Cz = D$$

where $D = Ax_0 + By_0 + Cz_0$

For an arbitrary dimension n it is analogously defined. In the general case it is cold a **hyperplane**. *For the case* $n = 2$ *which geometric object is the hyperplane?*

Example 1

- The plane through the point $(3, 2, 1)$ with normal vector $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ has equation:

$$(2\mathbf{i} - \mathbf{j} + 4\mathbf{k}) \cdot ((x-3)\mathbf{i} + (y-2)\mathbf{j} + (z-1)\mathbf{k}) = 0 \iff$$
$$\iff 2(x-3) - (y-2) + 4(z-1) = 0 \iff 2x - y + 4z = 8$$

Example 2

Given the plane with equation

$$7x + 2y - 3z = 1$$

find a normal vector to the plane and identify three points that lie on that plane.

- A possible normal vector is $\mathbf{n} = 7\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.
- However, any nonzero scalar multiple of $\mathbf{n}$ will work as well.
- Algebraically, the effect of using a scalar multiple of $\mathbf{n}$ as normal is to multiply by such a scalar the equation.

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

- Finding three points in the plane is not difficult. **How can we find it**?

From high school geometry, you may recall that

A plane is determined by
three (noncollinear) points

Example 4

Find an equation of the plane that contains the points $P_0(1,2,0)$, $P_1(3,1,2)$, and $P_2(0,1,1)$.

Two different approaches

- There are two ways to solve this problem
- The first approach is algebraic and rather uninspired.
- The second method of solution is cleaner and more geometric.

Example 4

A plane is determined by
three (noncollinear) points

Find an equation of the plane that contains the points $P_0(1, 2, 0)$, $P_1(3, 1, 2)$, and $P_2(0, 1, 1)$.

First approach

- Any plane must have an equation of the form

$$Ax + By + Cz = D$$

  for suitable constants A, B, C, and D.
- Thus, we need only to substitute the coordinates of P_0, P_1, and P_2 into this equation and solve for A, B, C, and D.

Example 4

A plane is determined by
three (noncollinear) points

Find an equation of the plane that contains the points $P_0(1,2,0)$, $P_1(3,1,2)$, and $P_2(0,1,1)$

First approach

$$Ax + By + Cz = D$$

- Substitution of P_0 gives $A + 2B = D$
- Substitution of P_1 gives $3A + B + 2C = D$
- Substitution of P_2 gives $B + C = D$

Example 4

A plane is determined by
three (noncollinear) points

Find an equation of the plane that contains the points $P_0(1, 2, 0)$, $P_1(3, 1, 2)$, and $P_2(0, 1, 1)$

First approach

- Hence, we must solve a system of 3 equations in 4 unknowns

$$\begin{cases} A + 2B = D \\ 3A + B + 2C = D \\ B + C = D \end{cases}$$

- In general, such a system has either no solution or else infinitely many solutions.

Example 4

A plane is determined by
three (noncollinear) points

Find an equation of the plane that contains the points $P_0(1,2,0)$, $P_1(3,1,2)$, and $P_2(0,1,1)$

First approach

- Hence, we must solve a system of 3 equations in 4 unknowns

$$\begin{cases} A + 2B = D \\ 3A + B + 2C = D \\ B + C = D \end{cases}$$

- We must be in the latter case, since we know that the three points P_0, P_1, and P_2 lie on some plane:

Some set of constants A, B, C, and D must exist.

Example 4

A plane is determined by three (noncollinear) points

Find an equation of the plane that contains the points $P_0(1,2,0)$, $P_1(3,1,2)$, and $P_2(0,1,1)$

First approach

- We can choose a value for one of A, B, C, or D, and then the other values will be determined.

Example 4

A plane is determined by
three (noncollinear) points

Find an equation of the plane that contains the points $P_0(1,2,0)$, $P_1(3,1,2)$, and $P_2(0,1,1)$

First approach

$$\begin{cases} A = -\frac{1}{7}D \\ 7C = 3D \\ B = \frac{4}{7}D \end{cases}$$

- Thus, if in we take $D = -7$ (for example), then $A = 1, B = -4, C = -3$, and the equation of the plane is

$$x - 4y - 3z = -7$$

Example 4

A plane is determined by
three (noncollinear) points

Find an equation of the plane that contains the points $P_0(1, 2, 0)$, $P_1(3, 1, 2)$, and $P_2(0, 1, 1)$

Second approach

- The idea is to make use of equation

$$\mathbf{n} \cdot \overrightarrow{P_0 P} = 0$$

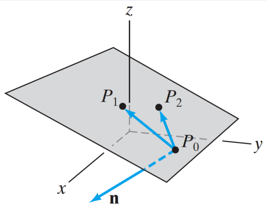
- Therefore, we need to know
  1. The coordinates of a particular point on the plane (no problem, we are given three such points).
  2. A vector $\mathbf{n}$ normal to the plane.

Example 4

A plane is determined by
three (noncollinear) points

Find an equation of the plane that contains the points $P_0(1,2,0)$, $P_1(3,1,2)$, and $P_2(0,1,1)$

Second approach



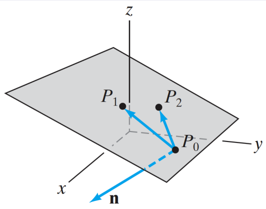
- The vectors $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P_2}$ both lie in the plane

Example 4

A plane is determined by
three (noncollinear) points

Find an equation of the plane that contains the points $P_0(1,2,0)$, $P_1(3,1,2)$, and $P_2(0,1,1)$

Second approach



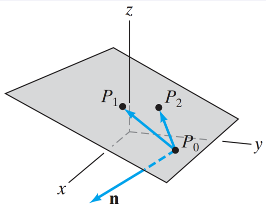
- In particular, the normal vector **n** must be perpendicular to both $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P_2}$

Example 4

A plane is determined by
three (noncollinear) points

Find an equation of the plane that contains the points $P_0(1, 2, 0)$, $P_1(3, 1, 2)$, and $P_2(0, 1, 1)$

Second approach



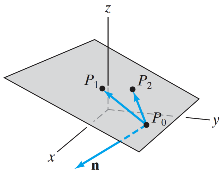
- Consequently, the cross product $\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}$ provides just what we need

Example 4

A plane is determined by
three (noncollinear) points

Find an equation of the plane that contains the points $P_0(1,2,0)$, $P_1(3,1,2)$, and $P_2(0,1,1)$

Second approach



$$\mathbf{n} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = (2\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \times (-\mathbf{i} - \mathbf{j} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 2 \\ -1 & -1 & 1 \end{vmatrix}$$

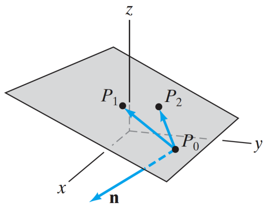
$$= \mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$$

Example 4

A plane is determined by
three (noncollinear) points

Find an equation of the plane that contains the points $P_0(1,2,0)$, $P_1(3,1,2)$, and $P_2(0,1,1)$

Second approach



- We take $P_0(1,2,0)$ to be the particular point in equation

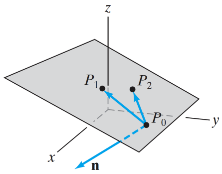
$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0$$

Example 4

A plane is determined by three (noncollinear) points

Find an equation of the plane that contains the points $P_0(1,2,0)$, $P_1(3,1,2)$, and $P_2(0,1,1)$

Second approach



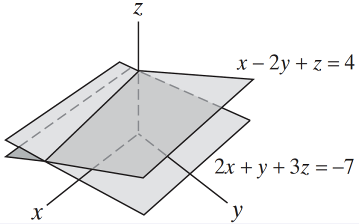
$$(\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}) \cdot ((x-1)\mathbf{i} + (y-2)\mathbf{j} + z\mathbf{k}) = 0$$

or
$$(x-1) - 4(y-2) - 3z = 0$$

Example 5

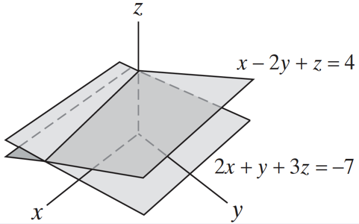
- Consider the two planes having equations

$$x - 2y + z = 4 \text{ and } 2x + y + 3z = -7$$



- Determine a set of parametric equations for their line of intersection

Example 5

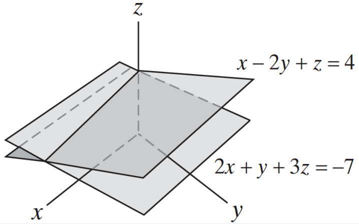


- We use Proposition 2.1

$$\mathbf{r}(t) = \mathbf{b} + t\mathbf{a}$$

- Thus, we need to find
  - A point on the line, and
  - A vector parallel to the line

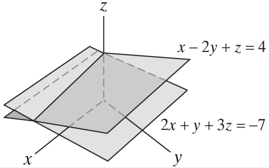
Example 5



- First, we find the coordinates (x, y, z) of a point on the line
- This coordinates must satisfy the system of simultaneous equations given by the two planes

$$\begin{cases} x - 2y + z = 4 \\ 2x + y + 3z = -7 \end{cases}$$

Example 5

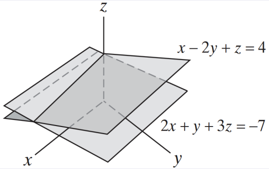


$$\begin{cases} x - 2y + z = 4 \\ 2x + y + 3z = -7 \end{cases}$$

- From the equations it is not too difficult to produce a single solution (x, y, z)
- For example, if we let $z = 0$ we obtain the simpler system

$$\begin{cases} x - 2y = 4 \\ 2x + y = -7 \end{cases}$$

Example 5



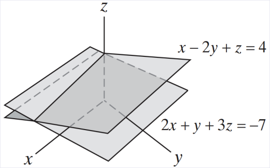
$$\begin{cases} x - 2y = 4 \\ 2x + y = -7 \end{cases}$$

- The solution to the system of equations is readily calculated to be:

$$x = -2, y = -3$$

- Thus, $(-2, -3, 0)$ are the coordinates of a point on the line.

Example 5



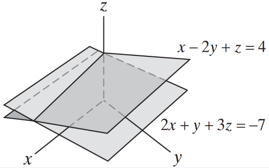
- Second, we find a vector parallel to the line of intersection
- Note that such a vector must be perpendicular to the two normal vectors to the planes (since the line is inside of both planes).
- The normal vectors to the planes are

$$\mathbf{i} - 2\mathbf{j} + \mathbf{k} \text{ and } 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$$

- A vector parallel to the line of intersection is given by

$$(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \times (2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) = -7\mathbf{i} - \mathbf{j} + 5\mathbf{k}$$

Example 5



- Hence, Proposition 2.1 implies that a vector parametric equation for the line is

$$\mathbf{r}(t) = (-2\mathbf{i} - 3\mathbf{j}) + t(-7\mathbf{i} - \mathbf{j} + 5\mathbf{k})$$

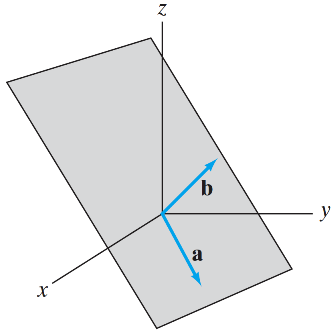
- And a standard set of parametric equations is

$$\begin{cases} x = -7t - 2 \\ y = -t - 3 \\ z = 5t \end{cases}$$

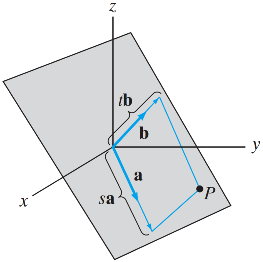
1. Equations for Planes and Distance Problems
    - Equations for Planes
    - Parametric Equation of the Plane
    - Distance Problems

Parametric Equations of Planes Through the Origin in $\mathbb{R}^3$

- Suppose that $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are two nonzero, nonparallel vectors in $\mathbb{R}^3$.
- Then, $\mathbf{a}$ and $\mathbf{b}$ determine a plane in $\mathbb{R}^3$ that passes through the origin.



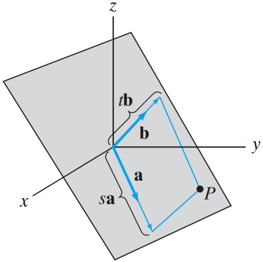
Parametric Equations of Planes Through the Origin in $\mathbb{R}^3$



- To find the coordinates of a point $P(x, y, z)$ in this plane, draw a parallelogram, i.e., that is a linear combination of the vectors **a** and **b**.
- So there exist scalars s and t such that the position vector of P is

$$\overrightarrow{OP} = s\mathbf{a} + t\mathbf{b}$$

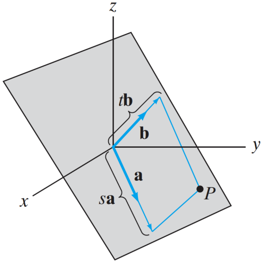
Parametric Equations of Planes Through the Origin in $\mathbb{R}^3$



- The plane may be described as

$$\Pi_0 \equiv \{\mathbf{x} \in \mathbb{R}^3 | \mathbf{x} = s\mathbf{a} + t\mathbf{b}; s, t \in \mathbb{R}\}$$

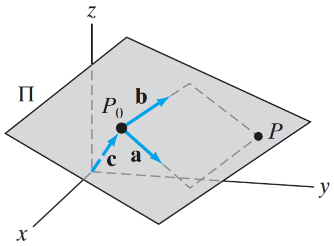
General Parametric Equations of Planes in $\mathbb{R}^3$



- Now, we seek to describe a general plane Π not necessarily passing through the origin.
- Let $\mathbf{c} = (c_1, c_2, c_3) = \overrightarrow{OP_0}$ denote the position vector of a particular point P_0 in Π.
- Let $\mathbf{a}$ and $\mathbf{b}$ be two (nonzero, nonparallel) vectors that determine the plane through the origin Π_0 parallel to Π.

General Parametric Equations of Planes in $\mathbb{R}^3$

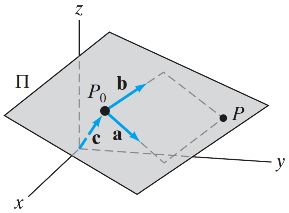
- We parallel translate **a** and **b** so that their tails are at the head of **c**



- The position vector of any point $P(x, y, z)$ in Π may be described as

$$\overrightarrow{OP} = s\mathbf{a} + t\mathbf{b} + \mathbf{c}$$

Proposition 5.1



$$\mathbf{x}(s,t) = s\mathbf{a} + t\mathbf{b} + \mathbf{c}$$

- By taking components in formula, we readily obtain a set of parametric equations for Π:

$$\Pi \equiv \begin{cases} x = sa_1 + tb_1 + c_1 \\ y = sa_2 + tb_2 + c_2 \\ z = sa_3 + tb_3 + c_3 \end{cases} \quad t, s \in \mathbb{R}$$

Parametric Lines vs Parametric planes

- We need to use one parameters t to describe a line

$$\Lambda \equiv \begin{cases} x = a_1 t + b_1 \\ y = a_2 t + b_2 \\ z = a_3 t + b_3 \end{cases} \quad t \in \mathbb{R}$$

A line is a one-dimensional object

- We need to use two parameters s and t to describe a plane

$$\Pi \equiv \begin{cases} x = s a_1 + t b_1 + c_1 \\ y = s a_2 + t b_2 + c_2 \\ z = s a_3 + t b_3 + c_3 \end{cases} \quad t, s \in \mathbb{R}$$

A plane is a two-dimensional object

Example 6

Find Π a set of parametric equations for the plane that passes through the point $(1, 0, -1)$ and is parallel to the vectors $3\mathbf{i} - \mathbf{k}$ and $2\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$

$$\mathbf{x}(s,t) = s\mathbf{a} + t\mathbf{b} + \mathbf{c}$$

- From formula, any point on the plane is specified by

$$\begin{aligned} \mathbf{x}(s,t) &= s(3\mathbf{i} - \mathbf{k}) + t(2\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}) + (\mathbf{i} - \mathbf{k}) \\ &= (3s + 2t + 1)\mathbf{i} + 5t\mathbf{j} + (2t - s - 1)\mathbf{k} \end{aligned}$$

- The individual parametric equation may be read off as

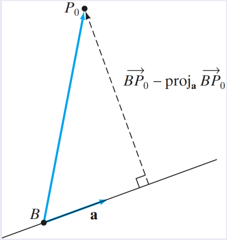
$$\begin{cases} x = 3s + 2t + 1 \\ y = 5t \qquad\qquad\qquad t, s \in \mathbb{R} \\ z = 2t - s - 1 \end{cases}$$

1. Equations for Planes and Distance Problems
   - Equations for Planes
   - Parametric Equation of the Plane
   - Distance Problems

Example 7. Distance between a point and a line

Find the distance between the point $P_0(2,1,3)$ and the line $\mathbf{l}(t) = t(-1,1,-2) + (2,3,-2)$ in two ways

Method 1



- From the vector parametric equations for the line, we read off
  - A point B on the line, namely, $B = (2,3,-2)$, and
  - A vector $\mathbf{a}$ parallel to the line, namely, $\mathbf{a} = (-1,1,-2)$

Example 7. Distance between a point and a line

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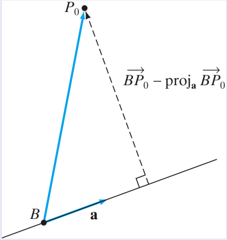
Method 1



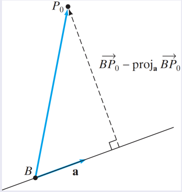
- The desired distance between P_0 and the line is provided by the length of the vector

$$\overrightarrow{BP_0} - \text{proj}_{\mathbf{a}} \overrightarrow{BP_0}$$

Example 7. Distance between a point and a line

Find the distance between the point $P_0(2,1,3)$ and the line $\mathbf{l}(t) = t(-1,1,-2) + (2,3,-2)$ in two ways

Method 1



$$\begin{aligned}
\overrightarrow{BP_0} &= (2,1,3) - (2,3,-2) = (0,-2,5) \\
\text{proj}_{\mathbf{a}}\overrightarrow{BP_0} &= \left(\frac{\mathbf{a}\cdot\overrightarrow{BP_0}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} = \left(\frac{(-1,1,-2)\cdot(0,-2,5)}{(-1,1,-2)\cdot(-1,1,-2)}\right)(-1,1,-2) \\
&= (2,-2,4)
\end{aligned}$$

Example 7. Distance between a point and a line

Find the distance between the point $P_0(2,1,3)$ and the line $\mathbf{l}(t) = t(-1,1,-2) + (2,3,-2)$ in two ways

Method 1

$$\begin{aligned} \overrightarrow{BP_0} &= (0,-2,5) \\ \text{proj}_{\mathbf{a}}\overrightarrow{BP_0} &= (2,-2,4) \end{aligned}$$

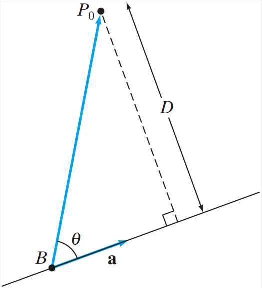
- The desired distance is

$$\begin{aligned} \|\overrightarrow{BP_0} - \text{proj}_{\mathbf{a}}\overrightarrow{BP_0}\| &= \|(0,-2,5) - (2,-2,4)\| \\ &= \|(-2,0,1)\| = \sqrt{5} \end{aligned}$$

Example 7. Distance between a point and a line

Find the distance between the point $P_0(2,1,3)$ and the line $\mathbf{l}(t) = t(-1,1,-2) + (2,3,-2)$ in two ways

Method 2

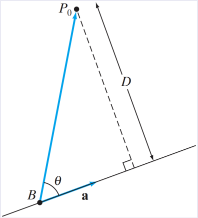


- In this case, we use a little trigonometry

Example 7. Distance between a point and a line

Find the distance between the point $P_0(2, 1, 3)$ and the line $\mathbf{l}(t) = t(-1, 1, -2) + (2, 3, -2)$ in two ways

Method 2



- If θ denotes the angle between the vectors $\mathbf{a}$ and $\overrightarrow{BP_0}$

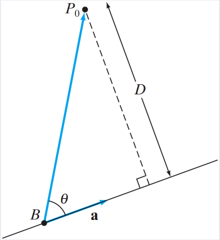
$$\sin\theta = \frac{D}{\|\overrightarrow{BP_0}\|}$$

where D denotes the distance between P_0 and the line

Example 7. Distance between a point and a line

Find the distance between the point $P_0(2,1,3)$ and the line $\mathbf{l}(t) = t(-1,1,-2) + (2,3,-2)$ in two ways

Method 2



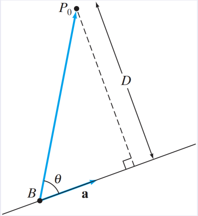
- Hence

$$D = \|\overrightarrow{BP_0}\| \sin\theta = \frac{\|\mathbf{a}\| \|\overrightarrow{BP_0}\| \sin\theta}{\|\mathbf{a}\|} = \frac{\|\mathbf{a} \times \overrightarrow{BP_0}\|}{\|\mathbf{a}\|}$$

Example 7. Distance between a point and a line

Find the distance between the point $P_0(2,1,3)$ and the line $\mathbf{l}(t) = t(-1,1,-2) + (2,3,-2)$ in two ways

Method 2



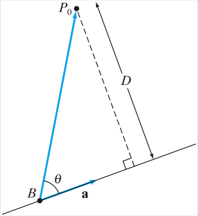
- Therefore, we calculate

$$\mathbf{a} \times \overrightarrow{BP_0} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -2 \\ 0 & -2 & 5 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$$

Example 7. Distance between a point and a line

Find the distance between the point $P_0(2,1,3)$ and the line $\mathbf{l}(t) = t(-1,1,-2) + (2,3,-2)$ in two ways

Method 2



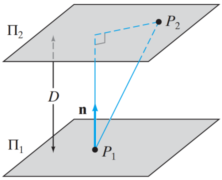
- So that the distance sought is

$$D = \frac{\|\mathbf{a} \times \overrightarrow{BP_0}\|}{\|a\|} = \frac{\|\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}\|}{\|-\mathbf{i} + \mathbf{j} - 2\mathbf{k}\|} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}$$

Example 8. Distance between parallel planes

Compute the distance between the parallel planes

$$\Pi_1 : 2x - 2y + z = 5 \quad \text{and} \quad \Pi_2 : 2x - 2y + z = 20$$

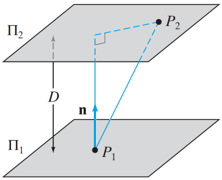


- The desired distance D is given by $\|\text{proj}_{\mathbf{n}} \overrightarrow{P_1 P_2}\|$ where
  - P_1 is a point on Π_1
  - P_2 is a point on Π_2, and
  - $\mathbf{n}$ is a vector normal to both planes

Example 8. Distance between parallel planes

Compute the distance between the parallel planes

$$\Pi_1 : 2x - 2y + z = 5 \quad \text{and} \quad \Pi_2 : 2x - 2y + z = 20$$



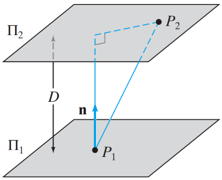
- The vector **n** that is normal to both planes may be read directly from the equation for either Π_1 or Π_2 as

$$\mathbf{n} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

Example 8. Distance between parallel planes

Compute the distance between the parallel planes

$$\Pi_1 : 2x - 2y + z = 5 \quad \text{and} \quad \Pi_2 : 2x - 2y + z = 20$$



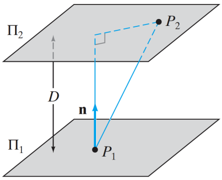
- It is not hard to find a point P_1 on Π_1, for instance, the point $P_1 = (0, 0, 5)$
- Similarly, take $P_2 = (0, 0, 20)$ for a point on Π_2
- Then

$$\overrightarrow{P_1 P_2} = (0, 0, 15)$$

Example 8. Distance between parallel planes

Compute the distance between the parallel planes

$$\Pi_1 : 2x - 2y + z = 5 \quad \text{and} \quad \Pi_2 : 2x - 2y + z = 20$$



$$\overrightarrow{P_1P_2} = (0,0,15) \quad \text{and} \quad \mathbf{n} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

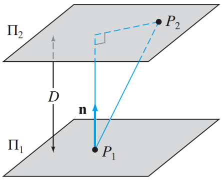
$$\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_2} = \left(\frac{\mathbf{n}\cdot\overrightarrow{P_1P_2}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = \left(\frac{(2,-2,-1)\cdot(0,0,15)}{(2,-2,1)\cdot(2,-2,1)}\right)(2,-2,1)$$

$$= -\frac{15}{9}(2,-2,1) = -\frac{5}{3}(2,-2,1)$$

Example 8. Distance between parallel planes

Compute the distance between the parallel planes

$$\Pi_1 : 2x - 2y + z = 5 \quad \text{and} \quad \Pi_2 : 2x - 2y + z = 20$$



$$\text{proj}_{\mathbf{n}} \overrightarrow{P_1 P_2} = -\frac{5}{3}(2, -2, 1)$$

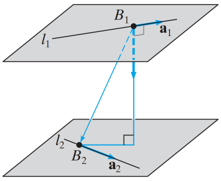
- Hence, the distance D that we seek is

$$D = \|\text{proj}_{\mathbf{n}} \overrightarrow{P_1 P_2}\| = \frac{5}{3}\sqrt{9} = 5$$

Example 9. Distance between two skew lines

Two lines in $\mathbb{R}^3$ are said to be **skew**
if they are neither intersecting nor parallel

- The lines must lie in parallel planes, and
- The distance between the lines is equal to the distance between the planes



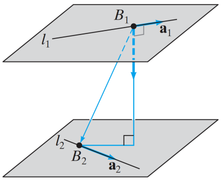
Find the distance between the two skew lines

$$\mathbf{l}_1(t) = t(2,1,3) + (0,5,-1) \quad \text{and} \quad \mathbf{l}_2(t) = t(1,-1,0) + (-1,2,0)$$

Example 9. Distance between two skew lines

Find the distance between the two skew lines

$$\mathbf{l}_1(t) = t(2,1,3) + (0,5,-1) \quad \text{and} \quad \mathbf{l}_2(t) = t(1,-1,0) + (-1,2,0)$$



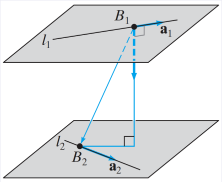
- We need to find the length of the projection of the vector between a point on each line onto a vector **n** that is
  - Perpendicular to both lines, and
  - Perpendicular to the parallel planes that contain the lines

$$\|\text{proj}_{\mathbf{n}} \overrightarrow{B_1 B_2}\|$$

Example 9. Distance between two skew lines

Find the distance between the two skew lines

$$\mathbf{l}_1(t) = t(2,1,3) + (0,5,-1) \quad \text{and} \quad \mathbf{l}_2(t) = t(1,-1,0) + (-1,2,0)$$



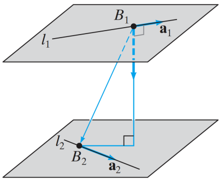
- From the vector parametric equations for the lines
  - Point $B_1(0,5,-1)$ is on the first line, and
  - Point $B_2(-1,2,0)$ is on the second line
  - Hence

$$\overrightarrow{B_1B_2} = (-1,2,0) - (0,5,-1) = (-1,-3,1)$$

Example 9. Distance between two skew lines

Find the distance between the two skew lines

$$\mathbf{l}_1(t) = t(2,1,3) + (0,5,-1) \quad \text{and} \quad \mathbf{l}_2(t) = t(1,-1,0) + (-1,2,0)$$



- For a vector $\mathbf{n}$ that is perpendicular to both lines, we may use

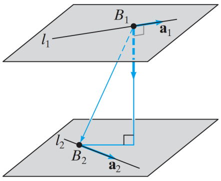
$$\mathbf{n} = \mathbf{a}_1 \times \mathbf{a}_2$$

  - $\mathbf{a}_1 = (2,1,3)$ is a vector parallel to the first line, and
  - $\mathbf{a}_2 = (1,-1,0)$ is a vector parallel to the second line

Example 9. Distance between two skew lines

Find the distance between the two skew lines

$$\mathbf{l}_1(t) = t(2,1,3) + (0,5,-1) \quad \text{and} \quad \mathbf{l}_2(t) = t(1,-1,0) + (-1,2,0)$$



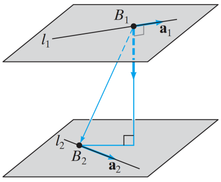
- $\mathbf{n} = \mathbf{a}_1 \times \mathbf{a}_2$
  - $\mathbf{a}_1 = (2,1,3)$ is a vector parallel to the first line, and
  - $\mathbf{a}_2 = (1,-1,0)$ is parallel to the second line

$$\mathbf{n} = \mathbf{a}_1 \times \mathbf{a}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{vmatrix} = 3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$$

Example 9. Distance between two skew lines

Find the distance between the two skew lines

$$\mathbf{l}_1(t) = t(2,1,3) + (0,5,-1) \quad \text{and} \quad \mathbf{l}_2(t) = t(1,-1,0) + (-1,2,0)$$



$$\mathbf{n} = 3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k} \quad \text{and} \quad \overrightarrow{B_1B_2} = (-1,-3,1)$$

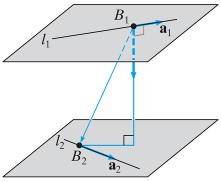
$$\text{proj}_{\mathbf{n}} \overrightarrow{B_1B_2} = \left(\frac{\mathbf{n} \cdot \overrightarrow{B_1B_2}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n} = \left(\frac{(-1,-3,1)\cdot(3,3,-3)}{(3,3,-3)\cdot(3,3,-3)} \right)(3,3,-3)$$

$$= -\frac{15}{27}(3,3,-3) = -\frac{5}{3}(1,1,-1)$$

Example 9. Distance between two skew lines

Find the distance between the two skew lines

$$\mathbf{l}_1(t) = t(2,1,3) + (0,5,-1) \quad \text{and} \quad \mathbf{l}_2(t) = t(1,-1,0) + (-1,2,0)$$



$$\text{proj}_{\mathbf{n}} \overrightarrow{B_1 B_2} = -\frac{5}{3}(1,1,-1)$$

- The desired distance is

$$\|\text{proj}_{\mathbf{n}} \overrightarrow{B_1 B_2}\| = \frac{5}{3}\sqrt{3}$$