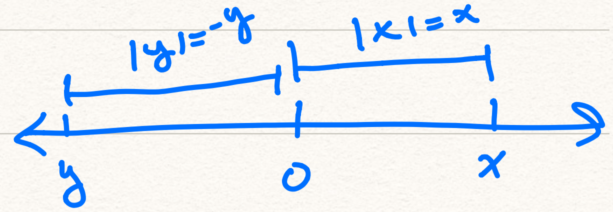
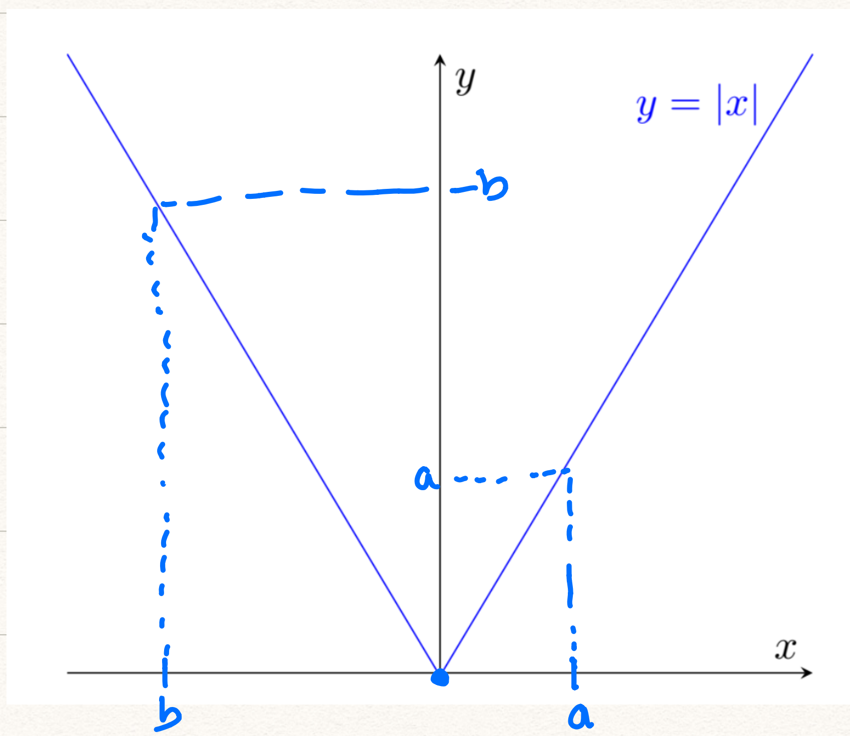
Let x be a real number. Its modulus (or absolute value) $|x|$ is defined by

$$|x| = \begin{cases} x & (x \geq 0) \\ -x & (x < 0) \end{cases}$$



It is sometimes useful to note that

$$|x| = \sqrt{x^2} \neq (\sqrt{x})^2$$



$\boxed{|x| \geq 0}$

$|x| = 0$

$\Rightarrow x = 0$

$\boxed{\text{Theorem}}$

For any real number x, $-|x| \le x \le |x|$.

Proof:

If $x \ge 0$, then $-|x| \le 0 \le x = |x|$

$$-|x| \le x \le |x|$$

If $x < 0$, then $-|x| = x < 0 \le |x|$

$$-|x| \le x \le |x|$$

$\boxed{\text{Theorem}}$

For any real numbers a and b, $|ab| = |a||b|$.

Proof:

$$|ab| = \sqrt{(ab)^2} = \sqrt{a^2}\sqrt{b^2} = |a|\,|b|$$

$\boxed{\text{Theorem: Triangle Inequality}}$

For any real numbers a and b,

$$|a+b| \leq |a| + |b|$$

$$|a+b| = \sqrt{(a+b)^2}$$

Proof:

$$|a+b|^2 = (a+b)^2 \qquad\qquad ab \leq |ab|$$

$$= a^2 + 2ab + b^2$$

$$= |a|^2 + 2ab + |b|^2$$

$$\leq |a|^2 + 2|ab| + |b|^2$$

$$= |a|^2 + 2|a||b| + |b|^2$$

$$= (|a| + |b|)^2$$

$$|a+b|^2 \leq (|a|+|b|)^2 \Rightarrow |a+b| \leq |a| + |b|$$

Remark: $|x|=0$ if and only if $x=0$.

$$x^2 = y^2$$

Problem 1.2 Prove that $|a+b|^2 = (|a|+|b|)^2$ if and only if $ab \geqslant 0$

$|x| = \sqrt{x^2}$

$\Longrightarrow$

$\Longleftarrow$

$\Longleftarrow$

$$|a+b|^2 = (a+b)^2$$
$$= a^2 + 2ab + b^2$$
$$= |a|^2 + 2ab + |b|^2 \qquad |x| = x$$
$$= |a|^2 + 2|ab| + |b|^2 \qquad |ab| = |a||b|$$
$$= |a|^2 + 2|a||b| + |b|^2$$
$$= (|a| + |b|)^2$$

If $ab \geq 0$ then $|a+b| = |a| + |b|$

$\Longrightarrow$

$$|a+b|^2 \underset{=(a+b)^2}{=} (|a| + |b|)^2$$
$$a^2 + 2ab + b^2 = |a|^2 + 2|a||b| + |b|^2$$
$$\cancel{|a|^2} + 2ab + \cancel{|b|^2} = \cancel{|a|^2} + 2|a||b| + \cancel{|b|^2}$$
$$2ab = 2|a||b| \geq 0$$
$$ab \geq 0$$

$|x| \geq 0$

$ab \geq 0$

$2ab \geq 0$

$a^2 + 2ab + b^2 \geq a^2 + b^2$

$(a+b)^2 \geq a^2 + b^2$

$\cancel{|a+b|^2 = |a|^2 + |b|^2}$

≥ 0

$a = -3 \qquad b = -2$

$ab = (-3)(-2) = 6$

$|a||b| = |-3||-2| = 3 \cdot 2 = 6$