Intuitively, the completeness of $\mathbb{IR}$ can be understood as follows:

If $\mathbb{IR}$ is represented as a straight line that extends indefinitely in both directions, the completeness of $\mathbb{IR}$ corresponds to the continuous nature of the line (solid line, no gaps).

Bounds

- A set $S \subseteq \mathbb{IR}$ is bounded above if there exists some $H \in \mathbb{IR}$ such that $x \leq H$ for every $x \in S$. If $H$ exists, it is called an upper bound of $S$. 

\[ \Longleftarrow \quad S \quad \text{\textcolor{black}{\text{\textcolor{blue}{\text{\textcolor{red}{\text{\Huge 1}}}}}}} \quad S \quad \rightarrow \quad \text{\textcolor{black}{\text{\textcolor{blue}{\text{\textcolor{red}{\text{\Huge 1}}}}}}} \quad \text{\textcolor{black}{\text{\textcolor{blue}{\text{\textcolor{red}{\text{\Huge H}}}}}}} \quad S \quad \rightarrow \quad S \]
A set $S \subseteq \mathbb{R}$ is bounded below if there exists some $h \in \mathbb{R}$ such that $h \leq x$ for every $x \in S$. If $h$ exists, it is called a lower bound of $S$.

A set which is both bounded below and above is just said to be bounded.

Examples:
(i) $\{1, 2, 3\}$ Bounded
    Bounded above: upper bounds $3, 10, 1000$
    bounded below: lower bounds $1, -10, -1000$

(ii) $\{x : 1 \leq x < 2\}$ \(1.9 = 2\)
    Bounded above: $2, 3, 50$\(, 1.9\) not upper bound

Bounded below: 1, 0, -12, -1M.

\((iii) \{ x : x > 0 \} \) < underbrace>{\text{not bounded above because for every } H \text{ in the set, } H+1 > H \text{ and } H+1 \text{ is also in the set}}\]

Bounded below: 0, -1, -15, -

Continuum property

- Every non-empty set of real numbers which is bounded above has a smallest upper bound. (supremum \( B = \sup S \))
- Every non-empty set of real numbers which is bounded below has a greatest lower bound. (infimum \( b = \inf S \))
Examples:
(i) \{1, 2, 3\}
   \text{Supremum: 3}
   \text{Infimum: 1}

(ii) \{x: 1 \leq x < 2\}
    \text{Infimum: 1}
    \text{Supremum: 2}

(iii) \{x: x > 0\}
     \text{Unbounded above} \Rightarrow \text{no supremum}
     \text{Infimum: 0}
     \underline{\text{Unbounded below}}
Maximum and minimum

If a non-empty set $S$ is bounded above and $B = \text{Sup}_S \in S$, then $B$ is called the maximum of $S$.

If a non-empty set $S$ is bounded below and $b = \text{Inf}_S \in S$, then $b$ is called the minimum of $S$. 
Examples:

(i) \{1, 2, 3\} Not an interval
   Maximum? yes 3
   Minimum? yes 1

(ii) \{x : 1 \leq x \leq 2\} Interval
    Maximum? No
    Minimum? yes 1

(iii) \{x : x > 0\} Interval
     Max? No, not bounded above.
     Min? No
Intervals

An interval $I$ is a set of real numbers with the property that, if $x \in I$ and $y \in I$ and $x \leq z \leq y$, then $z \in I$.

Notation:

- Bounded intervals:
  1. $(a, b) = \{ x : a < x < b \}$
  2. $[a, b] = \{ x : a \leq x \leq b \}$
  3. $[a, b) = \{ x : a \leq x < b \}$
  4. $(a, b] = \{ x : a < x \leq b \}$
Unbounded intervals:

5. $(a, \infty) = \{x : x > a\}$

6. $[a, \infty) = \{x : x \geq a\}$

7. $(-\infty, b) = \{x : x < b\}$

8. $(-\infty, b] = \{x : x \leq b\}$.

Example: Identify the set $A = \{x : |x| \leq 3\}$.

\[
\begin{array}{c}
\left[ \begin{array}{c}
-3 \\
0 \\
3 \\
\end{array} \right] \\
A = [-3, 3]
\end{array}
\]

Case I: if $x > 0$ then $|x| = x$

so $|x| \leq 3 \Rightarrow x \leq 3$

Case II: if $x \leq 0$, then $|x| = -x$

so $|x| \leq 3 \Rightarrow -x \leq 3$ or $-3 \leq x$

Putting case I and case II together we get

$-3 \leq x \leq 3$
Example: Identify the set

\[ B = \{ x : (x-1)(x-2)(x-3) < 0 \} = (-\infty, 1) \cup (2, 3) \]