▸ The induction principle is a property of the natural numbers that is used to prove statements that are true for all $n \in \mathbb{N}$.

It is quite useful when the statement can be expressed recursively.

Example $P(n): \; a_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$$a_{n+1} = \sum_{k=1}^{n+1} k = \underbrace{1+2+3+\cdots+n}_{a_n = \sum_{k=1}^{n} k} + n+1$$

$$a_n = \sum_{k=1}^{n} k = 1+2+3+\cdots+n$$

$$a_{n+1} = a_n + (n+1)$$

$\boxed{\text{Principle of Induction}}$

Suppose that for each $n \in \mathbb{N}$, $P(n)$ is a statement about the natural number n.

Suppose also that

- 1. $P(1)$ is true.
- 2. If $P(n)$ is true, then $P(n+1)$ is also true.

Then $P(n)$ is true for every $n \in \mathbb{N}$.

Example. For each $n \in \mathbb{N}$, $P(n)$: $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

$\boxed{\text{P(1) is true}}$

$n=1$: On one hand, $1+2$

$$\sum_{k=1}^{1} k = 1$$

On the other hand

$$\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = 1$$

$P(1)$ is true $\qquad$ I.H —— $P(n+1)$

$\boxed{\text{If } P(n) \text{ is true, then } P(n+1) \text{ is true}}$

Now, assume that $P(n)$ is true (I.H).

We need to prove that $P(n+1)$ is true.

$$P(n+1): \sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$$

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) \overset{\text{I.H}}{=} \frac{n(n+1)}{2} + (n+1)$$

$$= \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

Therefore, by induction,

$P(n)$ is true for all $n \in \mathbb{N}$.

Example. If $x \neq 1$, then

$$P(n): \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} \qquad \forall n \in \mathbb{N}.$$

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

1. On one hand, $\sum_{k=0}^{1} x^k = x^0 + x = 1 + x$

On the other hand, $\frac{1-x^2}{1-x} = \frac{(\cancel{1-x})(1+x)}{\cancel{1-x}} = 1+x$

So $P(1)$ is true.

2. Assume that $P(n)$ is true.

I need to show that $P(n+1)$ is true.

$$P(n+1): \sum_{k=0}^{n+1} x^k = \frac{1-x^{n+2}}{1-x}$$

$$\sum_{k=0}^{n+1} x^k = \sum_{k=0}^{n} x^k + x^{n+1} \overset{I.H}{=} \frac{1-x^{n+1}}{1-x} + x^{n+1}$$

$$= \frac{1-x^{n+1} + (1-x)x^{n+1}}{1-x} = \frac{1-\cancel{x^{n+1}}+\cancel{x^{n+1}}-x^{n+2}}{1-x}$$

$$= \frac{1-x^{n+2}}{1-x}$$

So $P(n+1)$ is true.

Therefore, by induction, $P(n)$ is true.

$$\sum_{k=0}^{n+1} x^k = \underbrace{x^0 + x^1 + x^2 + \cdots + x^n}_{} + x^{n+1}$$

$$= \sum_{k=0}^{n} x^k + x^{n+1}$$

Example For all $n \in \mathbb{N}$ $\qquad \mathbb{N} = \{1, 2, 3, \ldots\}$

$$a^n - b^n = (a-b) \sum_{k=1}^{n} a^{n-k} b^{k-1}$$

$$a^2 - b^2 = (a-b)(a+b)$$

$$a^3 - b^3 = (a-b) \sum_{k=1}^{3} a^{3-k} b^{k-1} = (a-b)(a^2 + ab + b^2)$$

$\boxed{P(1)?}$ On one hand, $a^1 - b^1$

On the hand, $(a-b) \sum_{k=1}^{1} a^{1-k} b^{k-1} = (a-b)$

So $P(1)$ is true.

$\boxed{\text{Induction step}}$

I.H: $a^n - b^n = (a-b) \sum_{k=1}^{n} a^{n-k} b^{k-1}$ is true

N.T.S: $\underbrace{a^{n+1} - b^{n+1} = (a-b)\sum_{k=1}^{n+1} a^{n+1-k} b^{k-1}}_{P(n+1)}$ is true.

$$(a-b)\sum_{k=1}^{n+1} a^{n+1-k} b^{k-1} =$$

$$(a-b)\left[\sum_{k=1}^{n} a^{n+1-k} b^{k-1} + a^{n+1-(n+1)} b^{(n+1)-1}\right]$$

$$= (a-b)\sum_{k=1}^{n} \underbrace{a^{n+1-k} b^{k-1}}_{\sum a\, a^{n-k} b^{k-1}} + (a-b)b^n$$

$$= a\underbrace{(a-b)\sum_{k=1}^{n} a^{n-k} b^{k-1}}_{I.H:\ a^n - b^n} + (a-b)b^n$$

$$= a(a^n - b^n) + (a-b)b^n = a^{n+1} - \cancel{ab^n} + \cancel{ab^n} - b^{n+1}$$

$$= a^{n+1} - b^{n+1}$$

Example. Prove by induction

$$1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}} > \sqrt{n} \qquad n>1$$

$n=2$

$2>1 \Rightarrow \sqrt{2}>\sqrt{1} \Rightarrow \sqrt{2}+1>\sqrt{1}+1$

$$P(2):\ 1+\frac{1}{\sqrt{2}} = \frac{\sqrt{2}+1}{\sqrt{2}} > \frac{\sqrt{1}+1}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}} = \sqrt{2}$$

So $P(2)$ is true.

Induction Step:

$$\text{IH: } 1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > \sqrt{n} + \frac{1}{\sqrt{n+1}} \text{ is true}$$

$$\text{NTS: } 1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}} > \sqrt{n+1} \text{ is true}$$

$$1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}} \overset{I.H}{>} \sqrt{n}+\frac{1}{\sqrt{n+1}}$$

$$= \frac{\sqrt{n(n+1)}+1}{\sqrt{n+1}}$$

$$n^2+n > n^2 \quad \longleftarrow \quad = \frac{\sqrt{n^2+n}+1}{\sqrt{n+1}}$$

$$\Downarrow$$

$$\sqrt{n^2+n} > \sqrt{n^2}$$

$$\sqrt{n^2+n}+1 > \sqrt{n^2}+1 \qquad > \frac{\sqrt{n^2}+1}{\sqrt{n+1}}$$

$$= \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}$$

So $P(n+1)$ is true and we have proved the desired result by induction.