Problems

Problem 6.1

(a) Prove that if f is continuous then so is $|f|$. Show that the reciprocal is false by finding a counterexample.

(b) What can be said about a function that is continuous but all the values it takes are in $\mathbb{Q}$?

Problem 6.2

(a) Let $f : [0,1] \to [0,1]$ be a continuous, surjective function. Prove that there exists $c \in [0,1]$ such that $f(c) = c$.

(b) Let f be a continuous function in $[a,b]$ and let $x_1, \ldots, x_n \in [a,b]$. Prove that there exists $c \in [a,b]$ such that $f(c) = \frac{1}{n}\sum_{k=1}^{n} f(x_k)$.

Problem 6.3 Consider the function

$$f(x) = \frac{1}{\lambda x^2 - 2\lambda x + 1}.$$

Determine for which values $\lambda \in \mathbb{R}$ the function is continuous in (a) $\mathbb{R}$, or (b) $[0,1]$.

Problem 6.4 Study the continuity of the following functions:

(i) $f(x) = \dfrac{e^{-5x} + \cos x}{x^2 - 8x + 12}$;

(ii) $f(x) = e^{3/x} + x^3 - 9$;

(iii) $f(x) = x^3 \tan(3x+2)$;

(iv) $f(x) = \sqrt{x^2 - 5x + 6}$;

(v) $f(x) = (\arcsin x)^3$;

(vi) $f(x) = (x-5)\log(8x-3)$;

(vii) $f(x) = x - \lfloor x \rfloor$;

(viii) $f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0; \end{cases}$

(ix) $f(x) = \begin{cases} \dfrac{\tan x}{\sqrt{x}}, & x > 0, \\ 0, & x = 0, \\ e^{1/x}, & x < 0; \end{cases}$

(x) $f(x) = \begin{cases} x, & x \in \mathbb{Q}, \\ -x, & x \notin \mathbb{Q}; \end{cases}$

(xi) $f(x) = \begin{cases} \sin(\pi x), & x < -1, \\ |x| - x, & -1 \leqslant x < 1, \\ (x-1)^3, & x \geqslant 1; \end{cases}$

(xii) $f(x) = \begin{cases} (x+1)^2, & x \leqslant -1, \\ \operatorname{sgn} x + 1, & -1 < x < 1, \\ 2x, & x \geqslant 1; \end{cases}$

(xiii) $f(x) = \begin{cases} x^2, & x \leqslant -2, \\ |x^2 - 1|, & -2 < x < 2, \\ 4x - 5, & x \geqslant 2; \end{cases}$

(xiv) $f(x) = \begin{cases} (x-1)^2, & x > 1, \\ x - \lfloor x \rfloor, & -1 \leqslant x \leqslant 1, \\ x + 1, & x < -1. \end{cases}$

Problem 6.5 Which of these equations have at least one solution in the specified set?:

(i) $x^2 - 18x + 2 = 0$, in $[-1,1]$;

(ii) $x - \sin x = 1$, in $\mathbb{R}$;

(iii) $e^x + 1 = 0$, in $\mathbb{R}$;

(iv) $\cos x + 2 = 0$, in $\mathbb{R}$;

(v) $f(x) = 0$, in $[-2,2]$, where f is given by
$$f(x) = \begin{cases} x^2 + 2, & -2 \leqslant x < 0, \\ -(x^2+2), & 0 \leqslant x \leqslant 2; \end{cases}$$

(vi) $\frac{1}{4}x^3 - \sin(\pi x) + 3 = \frac{7}{3}$, in $[-2,2]$;

(vii) $|\sin x| = \sin x + 3$, in $\mathbb{R}$.

Problem 6.6 Prove that any polynomial of odd degree has at least one real root.