Suppose that f is defined on an interral (a,b), except possibly for some point $\xi \in (a,b)$. We say that $f(x)$ tends (or converges) to a limit l as x tends to ξ and write

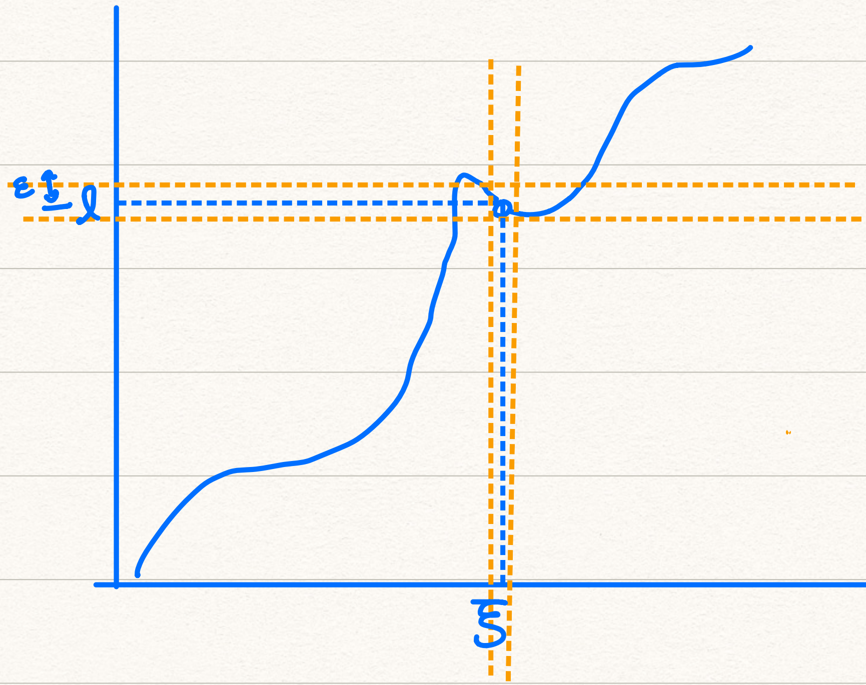
$$f(x) \to l \quad \text{as} \quad x \to \xi$$

or, alternatively,

$$\lim_{x \to \xi} f(x) = l$$

if the following criterion is satisfied.

> Given any $\varepsilon > 0$, we can find a $\delta > 0$ such that
>
> $|f(x) - l| < \varepsilon \qquad l - \varepsilon < f(x) < l + \varepsilon$
>
> provided that $0 < |x - \xi| < \delta$.


ε
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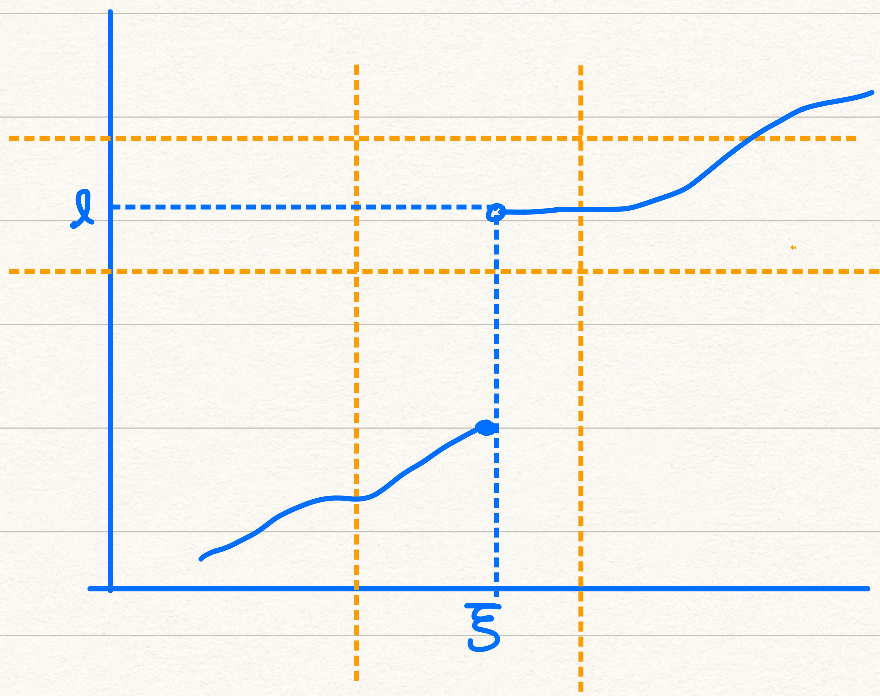
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Properties of Limits

Let f and g be defined on an interval (a,b) except possibly at $\xi \in (a,b)$. Suppose that $f(x) \to l$ and $g(x) \to m$ as $x \to \xi$ and Suppose that λ and μ are any real numbers. Then

1. $\lambda f(x) + \mu g(x) \longrightarrow \lambda l + \mu m$ as $x \to \xi$.
2. $f(x) g(x) \longrightarrow lm$ as $x \to \xi$.
3. $f(x)/g(x) \longrightarrow l/m$ as $x \to \xi$ (provided $m \neq 0$).

- For any polynomial $P(x)$

$$\lim_{x \to \xi} P(x) = P(\xi), \quad \forall \xi \in \mathbb{R}$$

- For any rational function $P(x)/Q(x)$

$$\lim_{x \to \xi} \frac{P(x)}{Q(x)} = \frac{P(\xi)}{Q(\xi)} \quad \forall \xi \in \mathbb{R} \text{ (provided } Q(\xi) \neq 0).$$

Example. Calculate the following limits.

a) $\lim_{x\to 1} \frac{x^2+4}{x^2-4}$ b) $\lim_{x\to 0} \frac{x^{73}+5x^{42}+9}{3x^{23}+7}$

c) $\lim_{x\to 0} \frac{\frac{1}{(1-x)^3} - 1}{x}$ d) $\lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a}$

For any function $g(x)\to 0$ as $x\to \xi$ we have

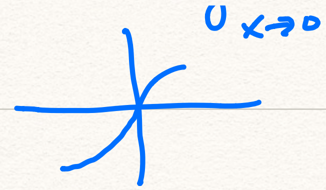
$$\lim_{x\to\xi} \frac{\sin g(x)}{g(x)} = 1, \quad \lim_{x\to\xi} \frac{1-\cos g(x)}{g(x)^2} = \frac{1}{2}$$

$$\lim_{x\to\xi} \frac{e^{g(x)}-1}{g(x)} = 1, \quad \lim_{x\to\xi} (1+g(x))^{1/g(x)} = e$$

$1^\infty \neq 1$

$g(x)\to 0$

a) $\lim\limits_{x\to 1} \dfrac{x^2+4}{x^2-4} = \dfrac{5}{-3} = \dfrac{-5}{3}$



b) $\lim\limits_{x\to 0} \dfrac{x^{73}+5x^{42}+9}{3x^{23}+7} = \dfrac{9}{7}$

$(a-b)^3 = a^3-3a^2b+3ab^2-b^3$

c) $\lim\limits_{x\to 0} \dfrac{\dfrac{1}{(1-x)^3} - 1}{x} = \lim\limits_{x\to 0} \dfrac{1-(1-x)^3}{x(1-x)^3}$

$= \lim\limits_{x\to 0} \dfrac{1-(1-3x+3x^2-x^3)}{x(1-x)^3} = \lim\limits_{x\to 0} \dfrac{3x-3x^2+x^3}{x(1-x)^3}$

$= \lim\limits_{x\to 0} \dfrac{\cancel{x}(3-3x+x^2)}{\cancel{x}(1-x)^3} = 3$

Geogebra

$(a-b)(a+b) = a^2-b^2$

d) $\lim\limits_{x \to a} \dfrac{\sqrt{x}-\sqrt{a}}{x-a} \cdot \dfrac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} = \lim\limits_{x \to a} \dfrac{\cancel{x-a}}{\cancel{(x-a)}(\sqrt{x}+\sqrt{a})}$

$= \lim\limits_{x \to a} \dfrac{1}{\sqrt{x}+\sqrt{a}} = \dfrac{1}{2\sqrt{a}}$

$$(x-a)(x+a) = x^2 + \cancel{ax} - \cancel{ax} - a^2$$
$$= x^2 - a^2$$

$$(x-a)^2 = (x-a)(x-a) = x^2 - ax - ax + a^2$$
$$= x^2 - 2ax + a^2$$
$$\neq x^2 - a^2$$

Examples

a) $\lim_{x\to 0} \dfrac{(\sin 2x^3)^2}{x^6}$

b) $\lim_{x\to 0} \dfrac{\tan x^2 + 2x}{x+x^2}$

c) $\lim_{x\to 0} \dfrac{\ln(1-2x)}{\sin x}$

d) $\lim_{x\to 0} \left(\dfrac{x}{\sin x}\right)^{\frac{\sin x}{\sin x - x}}$

e) $\lim_{x\to 0} (1+\sin x)^{2/x}$

f) $\lim_{x\to 0} \dfrac{1-\sqrt{1-x^2}}{x^2}$

One-sided limits

- We say that the left-handed limit of a function $f: A \to \mathbb{R}$ when x approaches ξ is ℓ, and denote it

$$\lim_{x\to\xi^-} f(x) = \ell \quad \text{or} \quad f(x) \to \ell \text{ as } x \to \xi$$

a) $\lim\limits_{x\to 0} \dfrac{(\sin 2x^3)^2}{x^6} = \dfrac{0}{0}$ $\pm$

$\lim\limits_{x\to \xi} \dfrac{\sin g(x)}{g(x)} = 1$

$\xi = 0 \quad g(x) = 2x^3$

$g \to 0$ when $x \to 0$

$= \lim\limits_{x\to 0} \left(\dfrac{2}{2} \dfrac{\sin 2x^3}{x^3} \right)^2$

$= \lim\limits_{x\to 0} \left(2 \cancelto{1}{\dfrac{\sin 2x^3}{2x^3}} \right)^2 = 2^2 = 4$

b) $\lim\limits_{x\to 0} \dfrac{\tan x^2 + 2x}{x + x^2} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0} \dfrac{\tan x^2 + 2x}{x(1+x)}$

$= \lim\limits_{x\to 0} \dfrac{\frac{\tan x^2}{x} + 2}{1+x} = \lim\limits_{x\to 0} \dfrac{\frac{\sin x^2}{x \cos x^2} + 2}{1+x}$

$= \lim\limits_{x\to 0} \dfrac{\frac{x}{x} \frac{\sin x^2}{x} \frac{1}{\cos x^2} + 2}{1+x} = \lim\limits_{x\to 0} \dfrac{\cancelto{0}{x} \cancelto{1}{\frac{\sin x^2}{x^2}} \cancelto{1}{\frac{1}{\cos x^2}} + 2}{1+x}$

$= 2$

if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - \ell| < \varepsilon$$

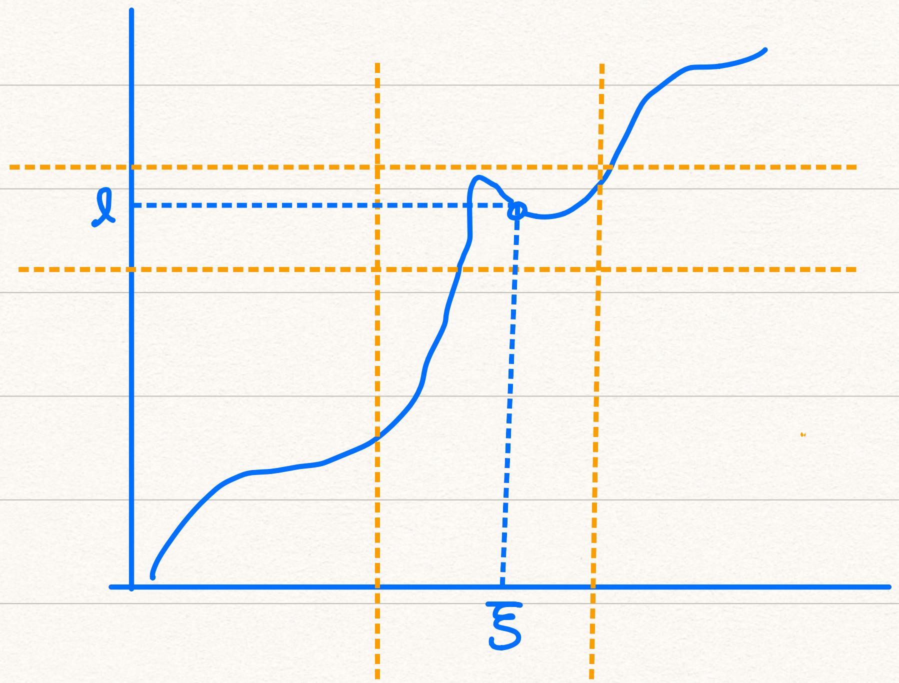
provided that $\xi - \delta < x < \xi$

- Similarly, we say that the right-handed limit of a function $f : A \to \mathbb{R}$ when x approaches ξ is ℓ, and denote it $\lim_{x \to \xi^+} f(x) = \ell$ or $f(x) \to \ell$ as $x \to \xi^+$

if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - \ell| < \varepsilon$$

provided that $\xi < x < \xi + \delta$.



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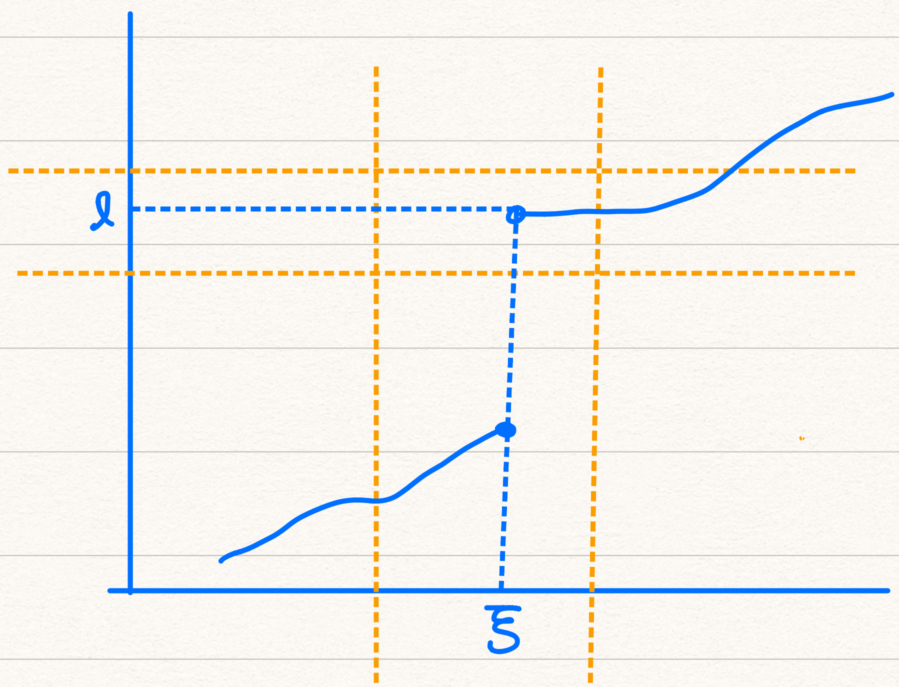


ℓ
ξ

Proposition

$$\lim_{x \to \xi} f(x) = \ell \iff \lim_{x \to \xi^-} f(x) = \lim_{x \to \xi^+} f(x) = \ell$$

Infinite Limits

We write sample definitions. The student will have little difficulty in supplying the definitions in other cases.

- We say that $f(x) \to +\infty$ as $x \to \xi^+$ if, given any $H > 0$, we can find a $\delta > 0$ such that

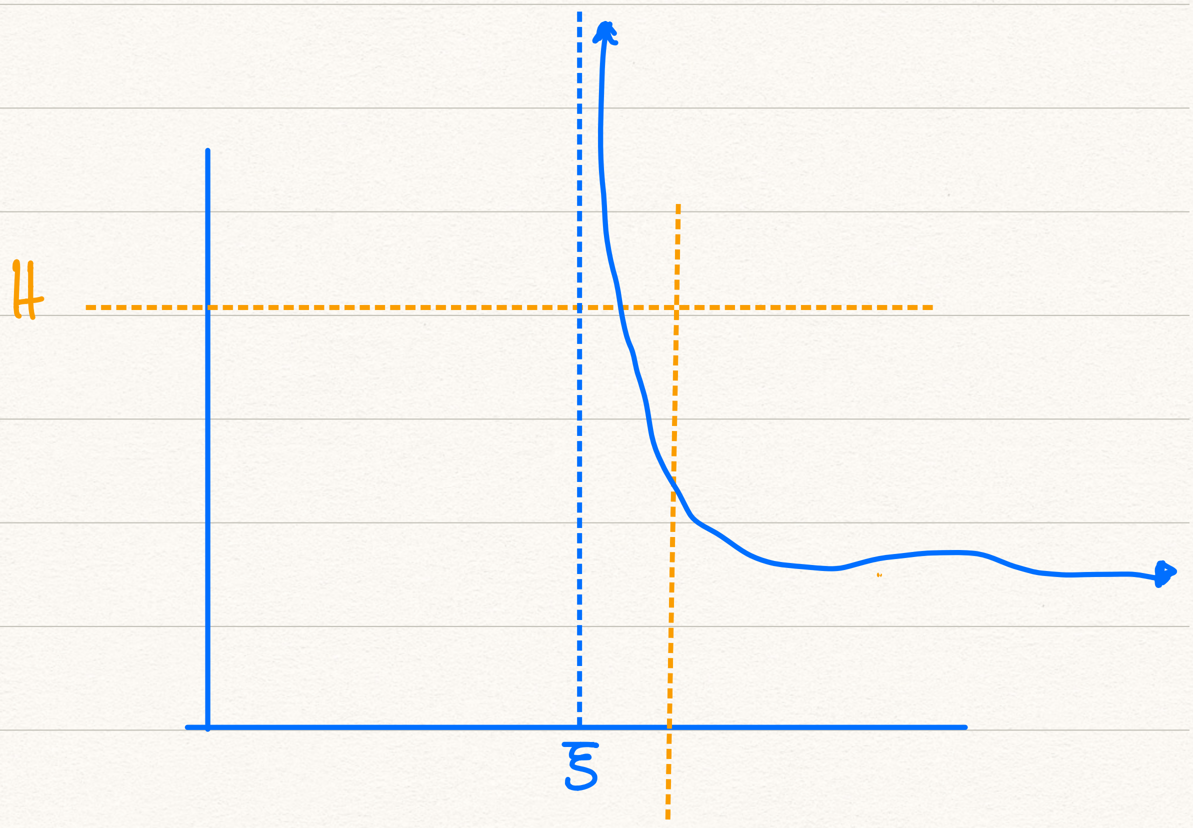
$$f(x) > H$$

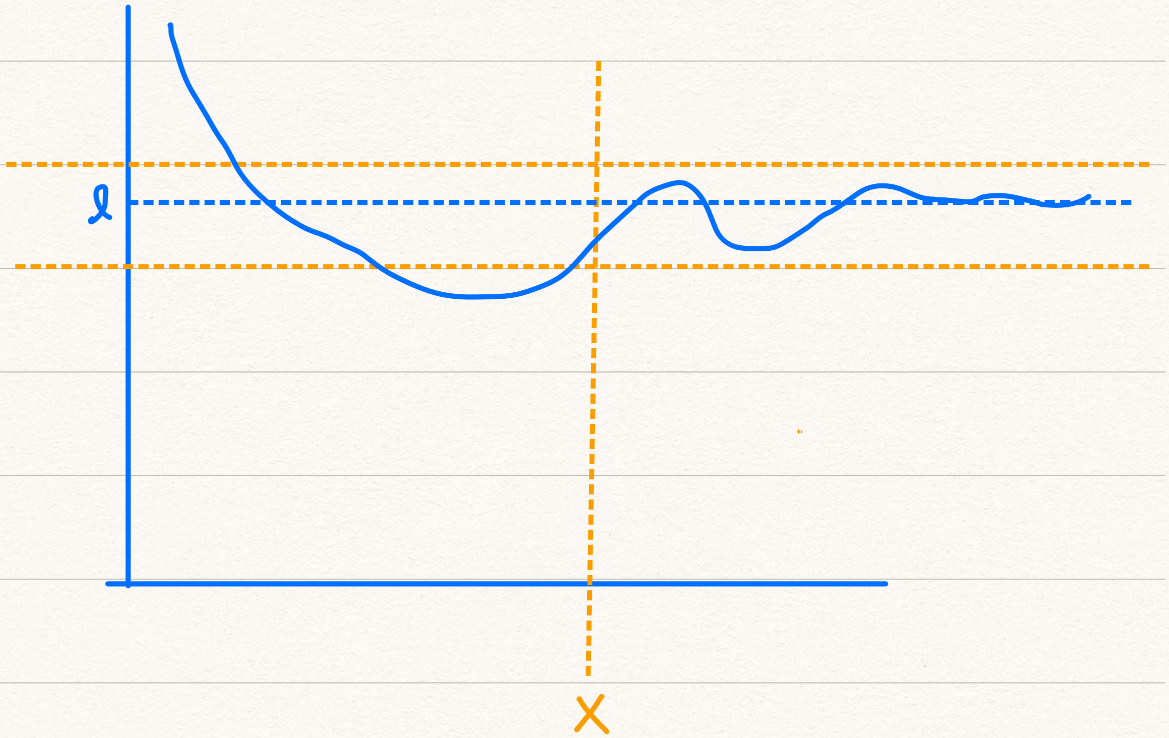
provided that $\xi < x < \xi + \delta$.

- We say that $f(x) \to \ell$ as $x \to \infty$ if, given any $\varepsilon > 0$, we can find an X such that

$$|f(x) - \ell| < \varepsilon$$

provided that $x > X$.





Examples.

a.) $\lim_{x \to \infty} \sin x$

b) $\lim_{x \to \infty} \dfrac{x^3+4x-7}{7x^2-\sqrt{2x^6+x^5}}$

c). $\lim_{x \to \pm\infty} \dfrac{x-2}{\sqrt{4x^2+1}}$

Sandwich Rule

If $\lim_{x \to \xi} g(x) = \lim_{x \to \xi} h(x) = l$, and if $g(x) \leq f(x) \leq h(x)$ holds for all x in an interval containing ξ, then $\lim_{x \to \xi} f(x) = l$

Example

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$

$$\lim_{x \to a} \sqrt{x} = \sqrt{a}$$