Problem 3.2 Given the following recurrent sequences, find the general term and compute their limit:

(i) $a_{n+1} = \dfrac{a_n + 1}{2}$, with $a_0 = 0$;

Problem 3.3 Calculate the following limits:

(iii) $\lim_{n\to\infty} n\left(\sqrt{n^2+1} - n\right)$;

Problem 3.4 Calculate the following limits:

(i) $\lim_{n\to\infty} \dfrac{n}{\pi} \sin n\pi$;

(iv) $\lim_{n\to\infty} n^{-3/n}$;

Problem 3.8 Calculate the limit

$$\lim_{n\to\infty} \sum_{k=1}^{3n} \frac{1}{\sqrt{n^2+k}}$$

using the sandwich rule.

HINT: Use the largest and smallest terms in the sum to bound the sum from above and from below, respectively.

Problem 3.9 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive terms such that $\lim_{n\to\infty}(a_n - n) = \ell$.

(a) Prove that $\lim_{n\to\infty} \dfrac{a_n}{n} = 1$.

Problem 4.3 Discuss, depending on the value of the parameter a in the given range, whether the following series converge or diverge:

(ii) $\sum_{n=1}^{\infty} \dfrac{n^n}{a^n n!}$, for $a > 0$;

(iii) $\sum_{n=1}^{\infty} \dfrac{n! e^n}{n^{n+a}}$, for any $a \in \mathbb{R}$;

(iv) $\sum_{n=1}^{\infty} \dfrac{a^n}{(1+a)(1+a^2)\cdots(1+a^n)}$, for $a \geqslant 0$.

Problem 4.4 Determine whether the following series are absolutely convergent, and if not, whether they converge conditionally:

(i) $\sum_{n=2}^{\infty} \dfrac{(-1)^n}{\log n}$;

(ii) $\sum_{n=1}^{\infty} \sin\left(\pi n + \dfrac{1}{n}\right)$;

(iii) $\sum_{n=1}^{\infty} (-1)^n \left(\arctan \dfrac{1}{n}\right)^2$;

(iv) $\sum_{n=1}^{\infty} (-1)^n (\arctan n)^2$;

(v) $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n^2-1} - n\right)$;

(vi) $\sum_{n=1}^{\infty} (-1)^n \log\left(\dfrac{n}{n+1}\right)$;

Problem 4.5 Sum the following series:

(i) $\sum_{n=0}^{\infty} \dfrac{3^{n+1} - 2^{n-3}}{4^n}$;

Problem 4.6 Obtain the sum of the series $\sum_{n=1}^{\infty} \dfrac{1}{n^3 + 3n^2 + 2n}$ by rewriting it as a telescopic series.

HINT: Expand the general term in elementary fractions.