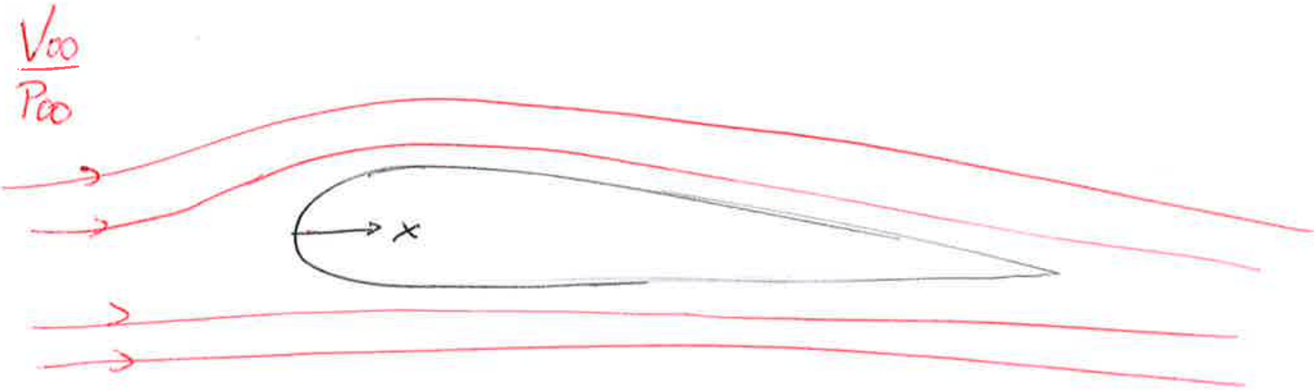
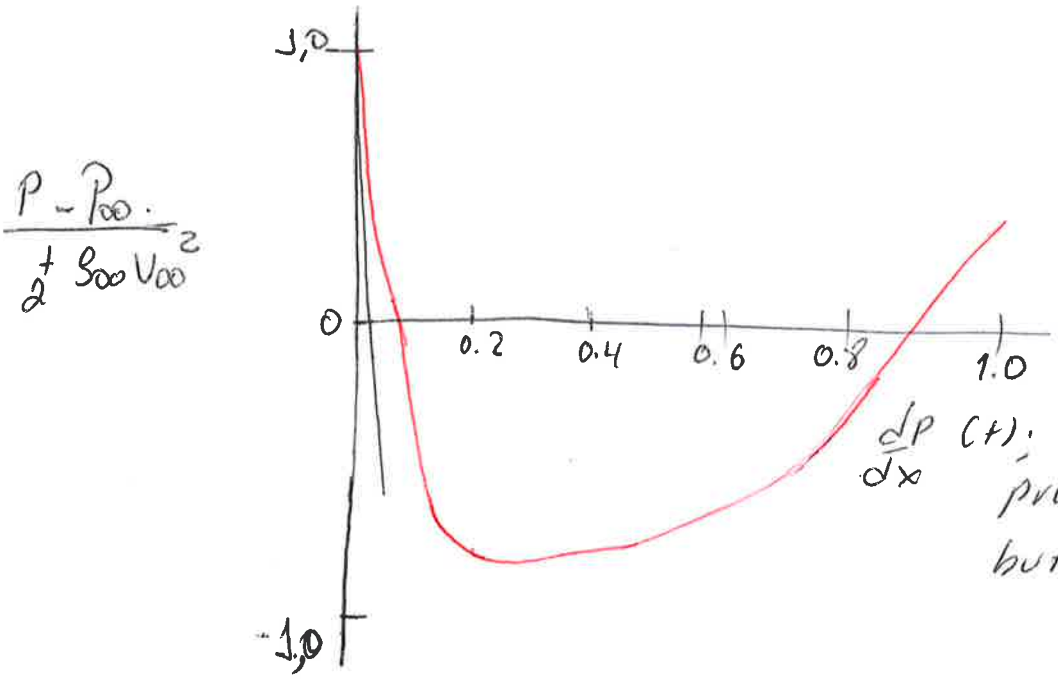
Flow separation



NACA LS(1) = 0417 airfoil

Angle of attack = 0°



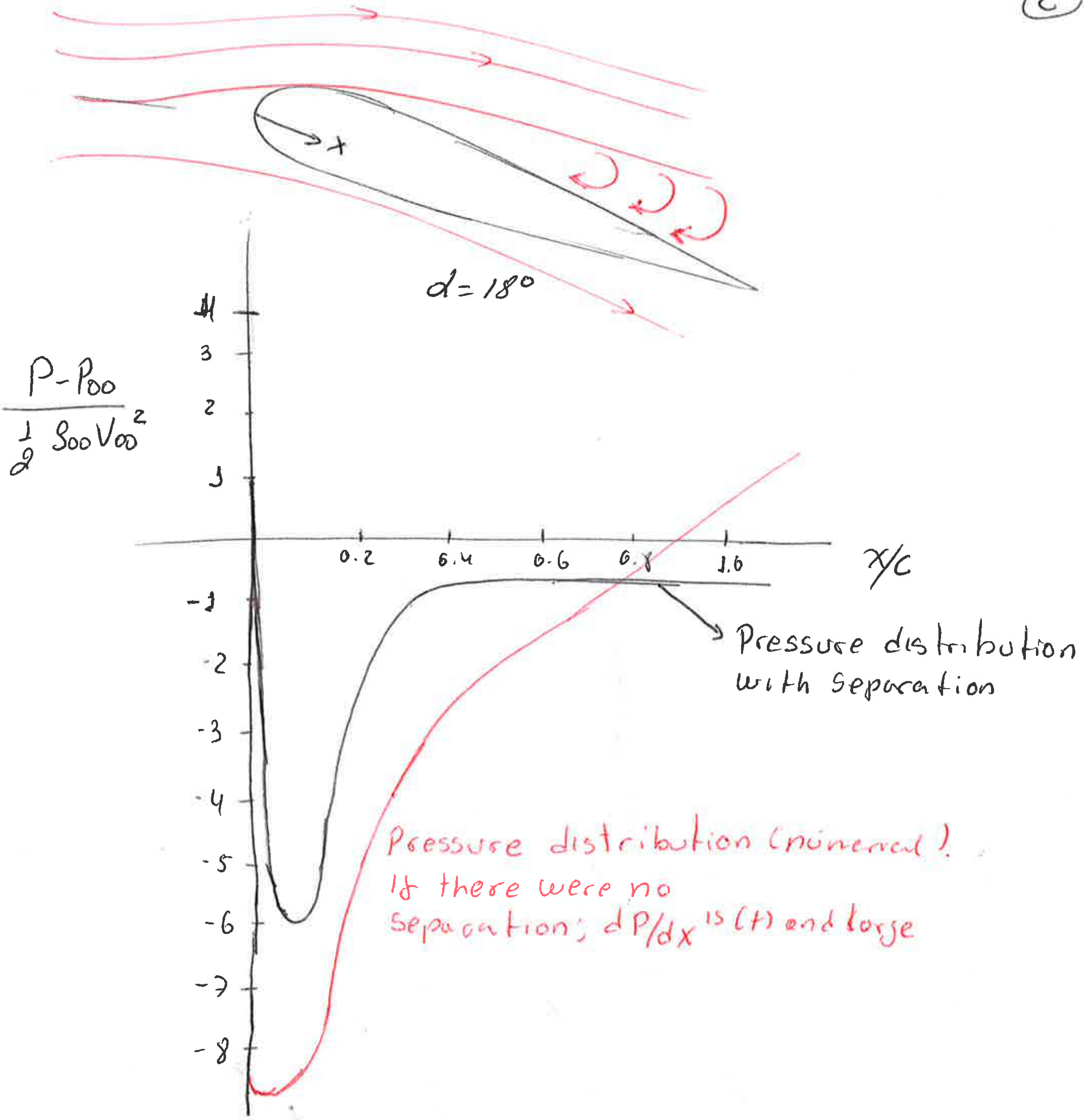
$\frac{dp}{dx}$ (+): this is an adverse pressure gradiente but is moderate.

Pressure distribution over the top for attached flow over an airfoil.

Pressure drang on airfoil is caused by the flow separation. For completely attached flow over an airfoil, the pressure acting on the rear surface gives rise to a force in the forword direction which completely counteract the pressure acting on the front surface producing a force rearward direction resulting zero pressure drag.

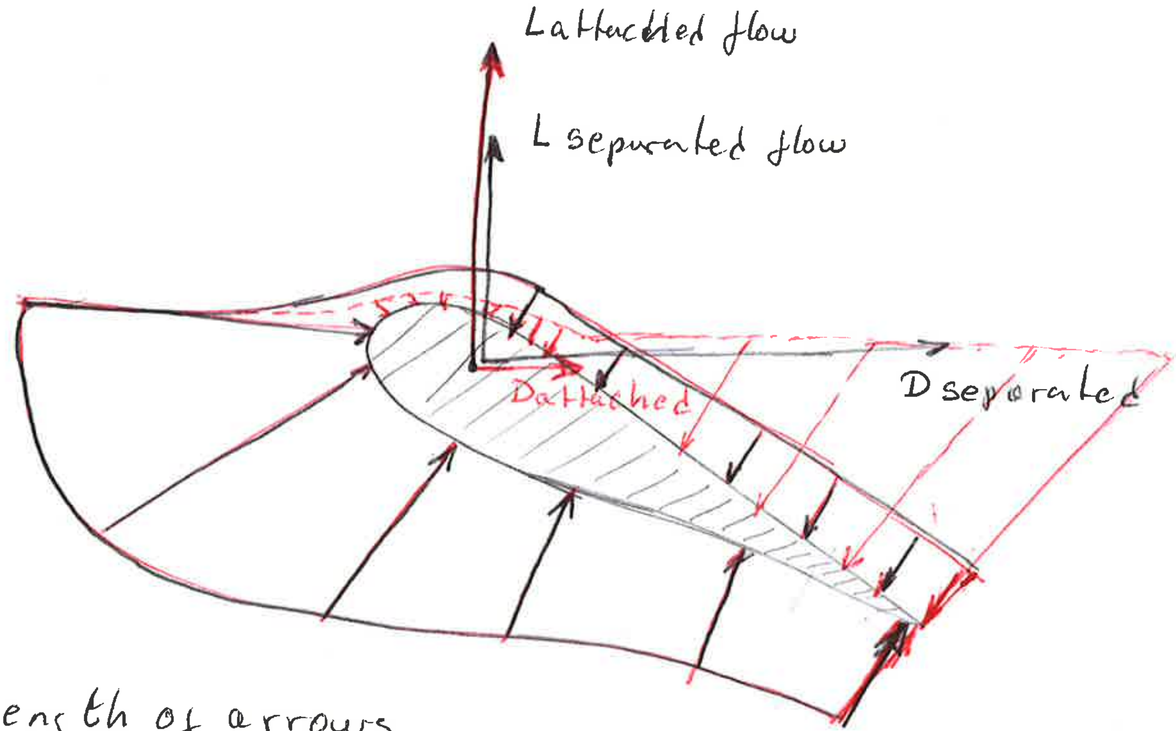
For a flow that is partially separated over the rear surface, the pressure on the rear surface pushing forword will be smaller than the fully attached case, and pressure drag is increased because the pressure acting on the front surface pushing backward is not counteracted.

The region where the pressure increases, and reaching a value slightly above the free stream pressure at the trailing edge. is called the Region of adverse Pressure gradient, this region is where the pressure increases in the flow direction.



The adverse pressure gradient would be severe: that is, dP/dx would be large. In such case, the real viscous flow tends to separate from the surface.

In this real separated flow, the actual surface pressure distribution is given by solid curve, obtained by a computational fluid dynamic viscous flow calculation using the complete Navier-Stokes equations. In comparison to the numerical solution, the actual pressure distribution does not dip to as low a pressure minimum and the pressure near the trailing edge does not recover to a value above P_∞.

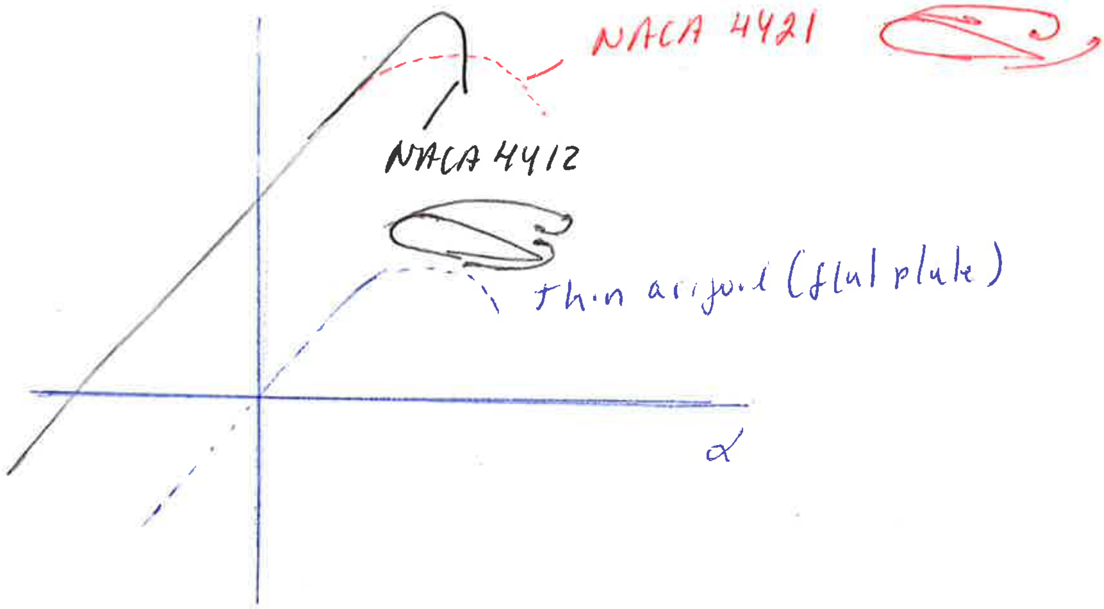


Length of arrows denoting pressure is proportional to $P - P_\infty$; where P_∞ is the P_{ref} and is arbitrary reference pressure slightly less than the minimum pressure on the airfoil.

Type of stalling phenomenon

Leading edge stalling: It is a characteristice relatively thin airfoils with thicuness ratios between 10 a 16 percent of the chord length. flow separation takes place rather suddenly and abruply over entire top surface of the airfoil, with the origin of this separation occurring at the leading edge.

Trailing edge stalling: this behavior is characteristic of the thicker airfoils such as the NACA 4421, there is a progressive and gradual movement of separation from trailing edge toward the leading edge as α is increased.



Aspects of airfoil aerodynamics

1. The lift-to-drag ratio L/D: An airfoil produces lift with minimumn of drag; L/D is a measure of the aerodynamic efficiency of an airfoil. L/D. ratio for a complete flight vehicle has a important impact on its flight performance.

2. the maximum lift coefficient C_{Lmax}: An effective airfoil produces a high value of C_{Lmax}. For a complete fligth vehicle, the maximum lift coefficient C_{Lmax} determines the stalling speed of aircraft.

$$V_{stall} = \sqrt{\frac{2W}{\rho_\infty S C_{Lmax}}}$$

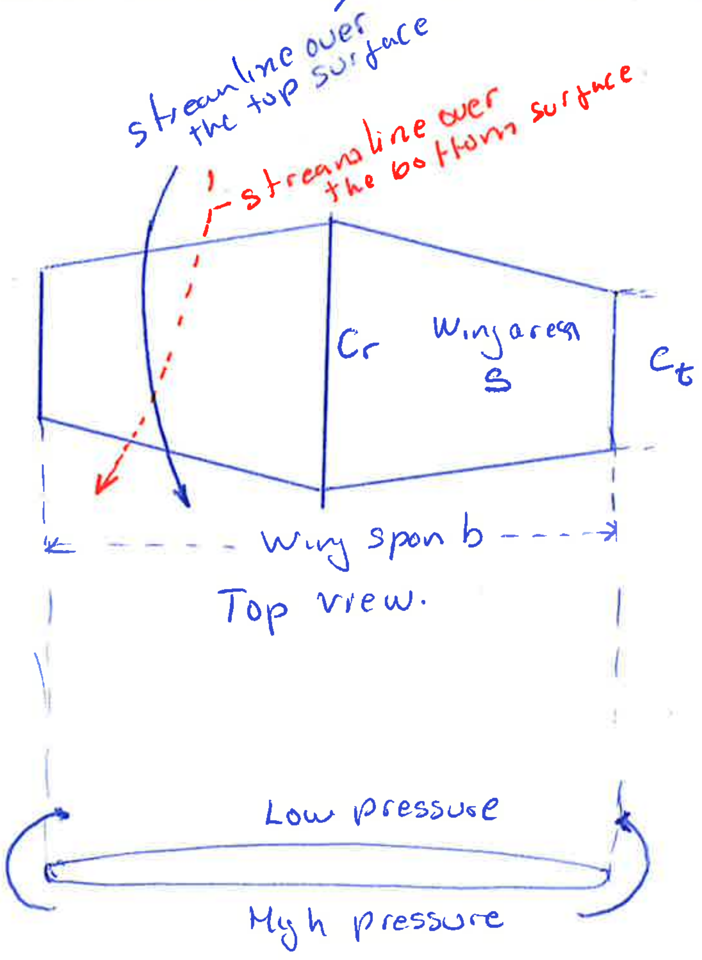
Incompressible Flow over Finite Wings

Introduction:

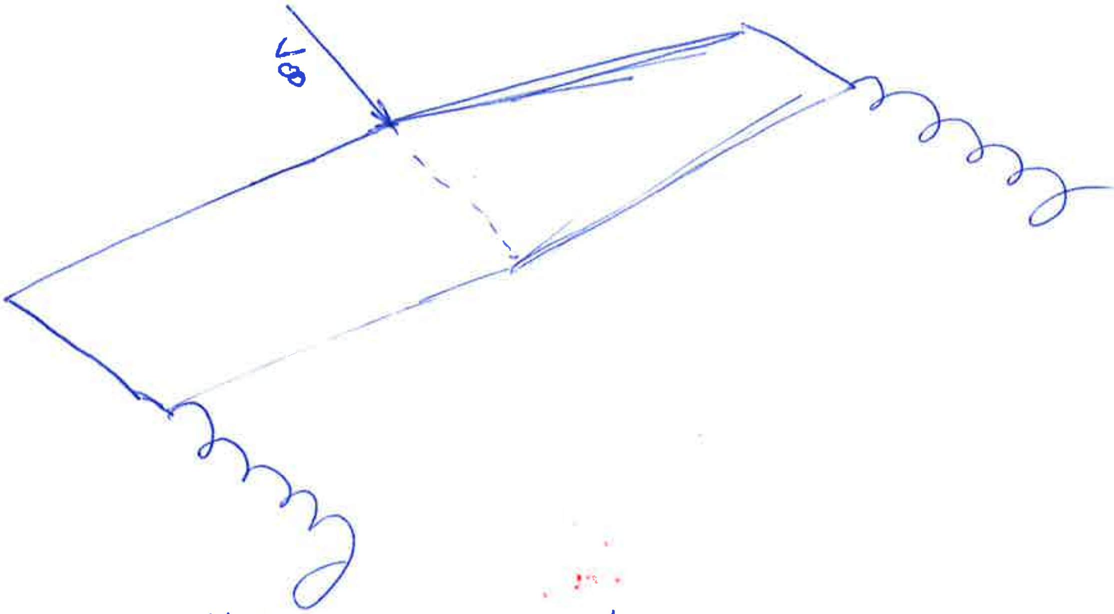
From momentum considerations, that a vortex which is stationary with respect to a uniform flow experience a force of magnitude $\rho V_\infty \Gamma$ in a direction perpendicular to V_∞. The resulting Kutta-Joukowski theorem states that the force experienced by unit span of a right cylinder of any cross section whatever is $\rho V_\infty \Gamma$ and it's directed perpendicular to V_∞. It can state that a stationary line vortex normal to a moving stream is the equivalent of an infinite span wing (an airfoil) as far as the resultant force is concerned.

The airfoil-vortex analogy also forms the basics for calculating the properties of the finite wing, since the lift (and therefore, the circulation) is zero at the tips of a finite wing and varies throughout the wing span, flow components appear that were not presente in the airfoil theory.

An airfoil is simply a section of a wing, and at first thought, you might expect the wing to behave exactly the same as the airfoil. However the flow over an airfoil is two-dimensional. In contrast, a finite wing is a three-dimensional body and the flow over a finite wing is three-dimensional; that is, there is a component of flow in the spanwise direction.



Front view.

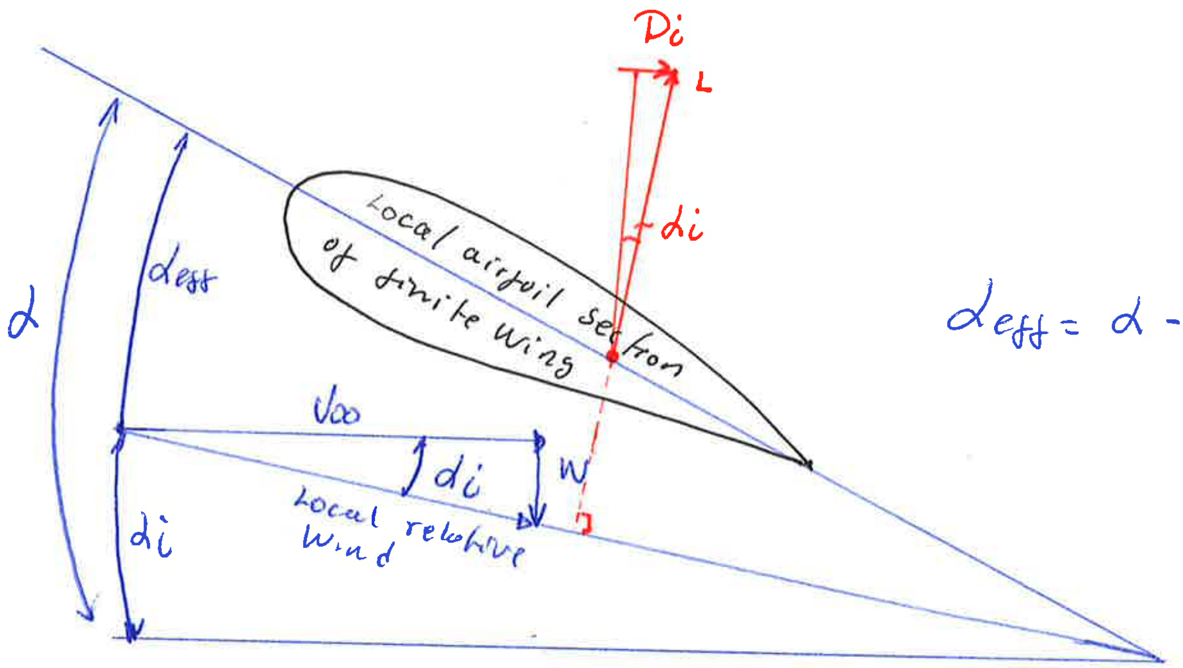


Wing - Tip vortices.

The tendency for the flow to "leak" around the wing tips has another important effect on the aerodynamics of the wing. This flow establishes a circulatory motion that trails downstream of the wing; that is, a trailing vortex is created at each wing tip. Tip vortices are essentially weak "Tornadoes" that trail downstream of the finite wing. These wing-tip vortices downstream of the wing induce a small downward component of air velocity in the neighborhood of the wing itself.

Two vortices tend to drag the surrounding air around with them. and this secondary movement induces a small velocity component in the downward direction of the wing.

This downward direction is called downwash. denoted by (w). In turn, the downwash combines with the freestream velocity V_∞ produce a local relative wind which is canted downward in the vicinity of each airfoil section wing



$$\alpha_{eff} = \alpha - \alpha_i$$

1. The angle of attack actually seen by the local airfoil section is the angle between the chord line and the local relative wind. α_{eff}, effective angle of attack.

$$\alpha_{eff} = \alpha - \alpha_i$$

2. The local lift vector is aligned perpendicular to the local relative wind. Consequently, there is a component of the local lift vector in direction of V_∞.

there is a Drag created by the presence of downwash. This drag is defined as induced drag.

1. The three-dimensional flow induced by the wing-tip vortices, simply alters the pressure distribution on the finite wing, a net pressure imbalance exits in the direction of V_∞. In this sense, induced drag is a type of pressure drag.

2. the wing-tip vortices contain a large amount of translational and rotational kinetic energy. Since the energy of the vortices serves no useful purpose, this power is essentially lost. so the extra power provided by the engine that goes into vortices is the extra power required from the engine to overcome the induced drag.

The total drag on a subsonic finite wing in real life is the sum of the induced D_i, the skin friction D_f and the pressure drag D_p, the latter two contributions come from the viscous effects. The sum of these two viscous-dominated drag is called profile drag.

$$C_d = \frac{D_f + D_p}{q_\infty S}$$

Profile drag coefficient for a finite wing. (moderate α)

and the induced drag coefficient as.

$$C_{Di} = \frac{D_i}{q_\infty S}$$

Total Drag coefficient for the finite wing is given by

$$C_D = C_d + C_{Di}$$

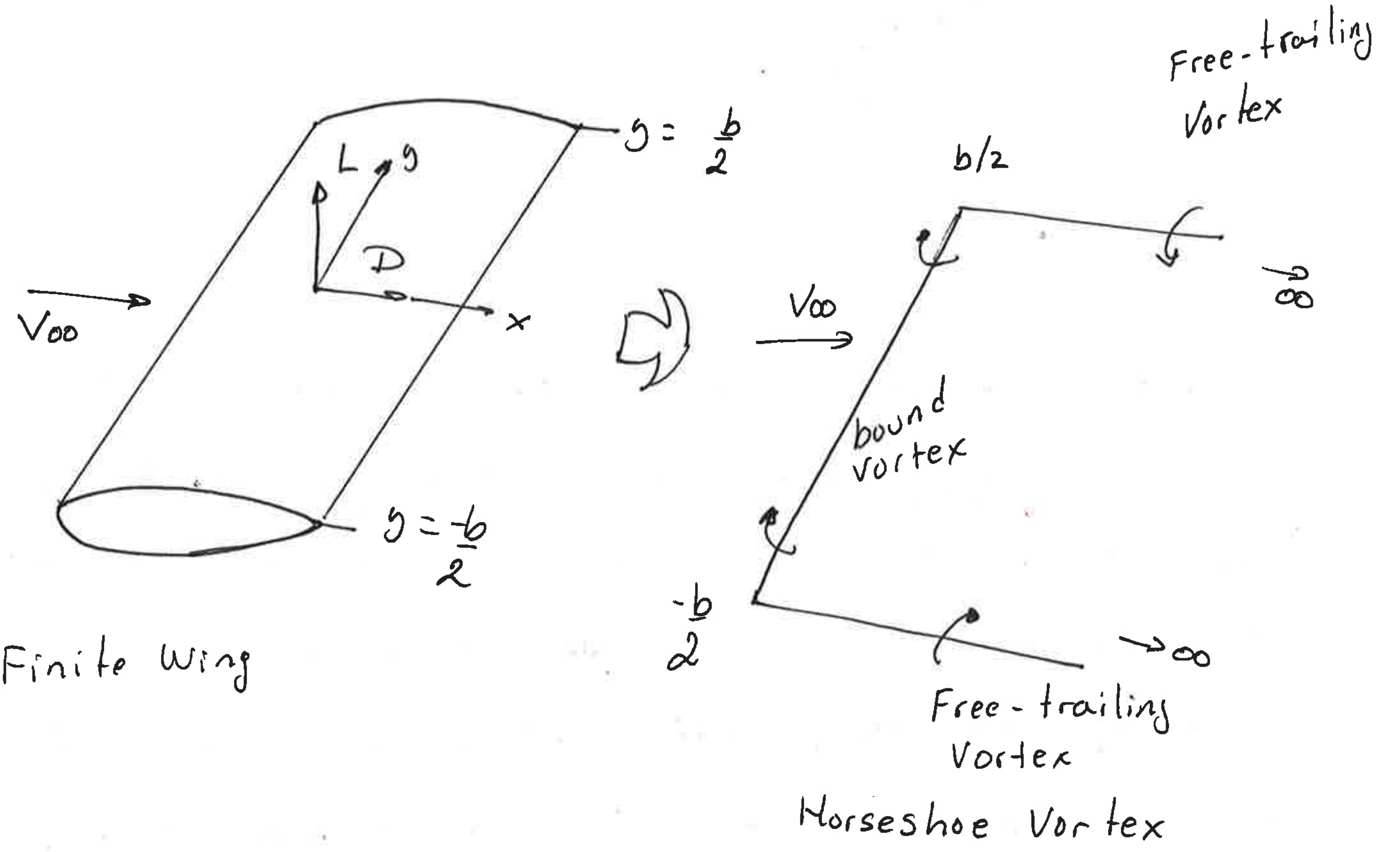
Usually C_d is obtained from airfoil data, such as an experimental data for profile drag coefficient. the Value of C_{Di} can be obtained from finite wind theory.

Prandtl's classical Lifting-Line theory

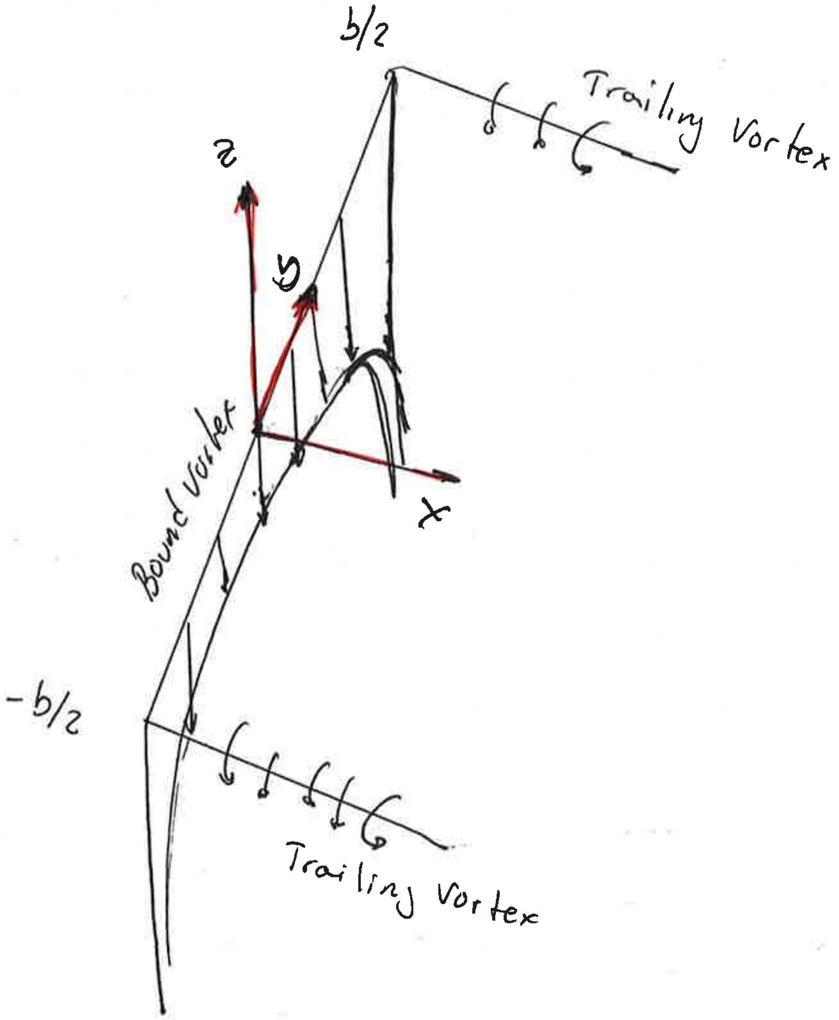
Ludwig Prandtl's was the first on developing a theory to predict the aerodynamic properties of finite wing.

He stated that a vortex filament of strength Γ that is somehow bound to a fixed location in a flow. (boundeVortex) will experience a Force.

$L = \rho_\infty V_\infty \Gamma$. This bound vortex is in contrast to a free vortex, which moves with the same fluid elements throughout a flow.

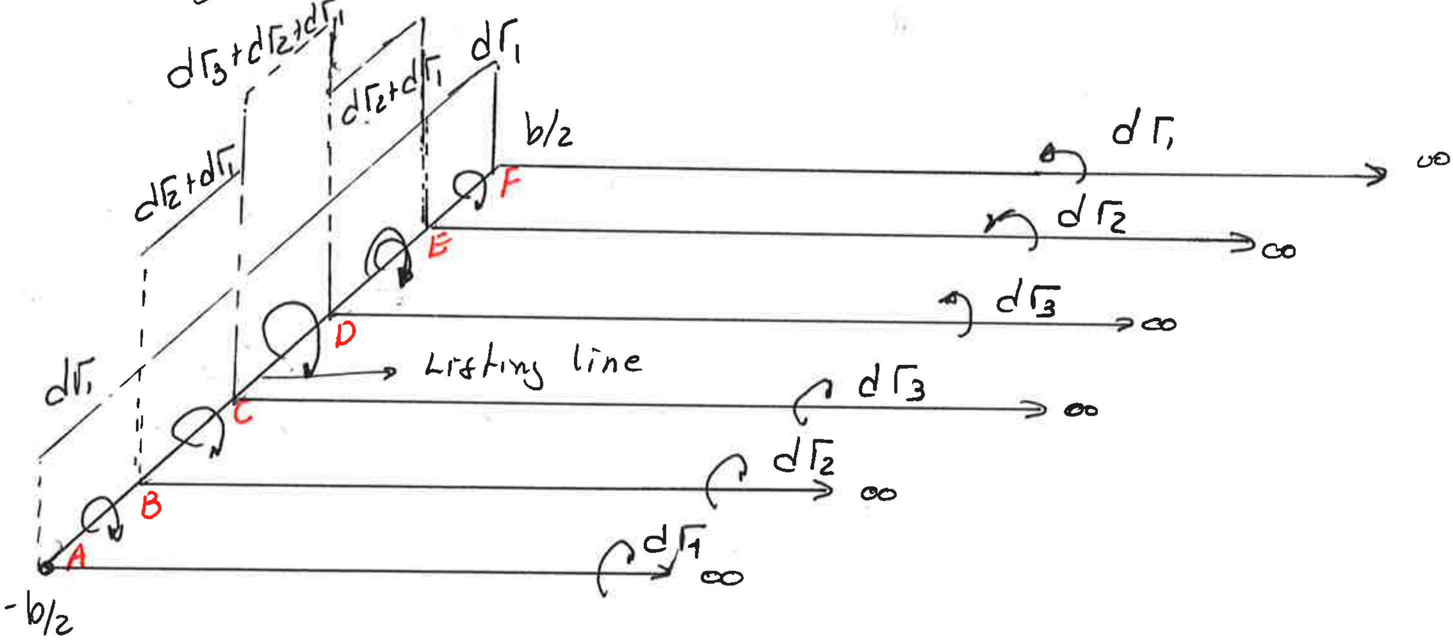


Consider the downwash w induced along the bound vortex from $-\frac{b}{2}$ to $\frac{b}{2}$ by the horseshoe vortex.



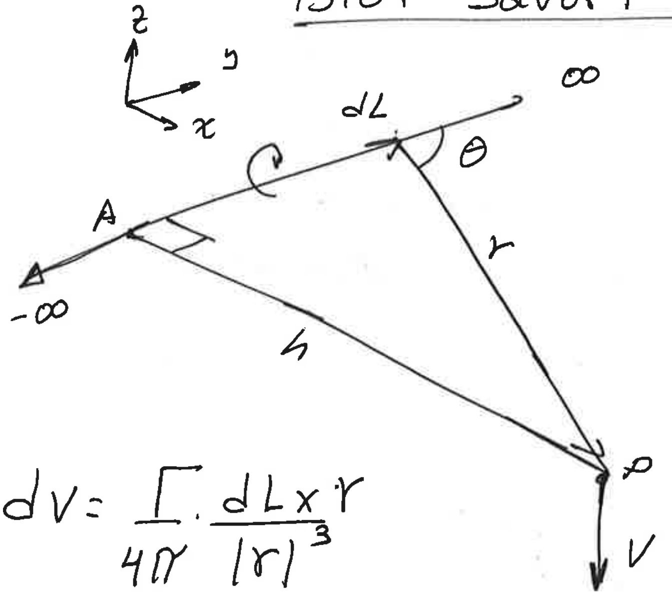
The two trailing vortices both contribute to induced velocity along the bound vortex, and these contributions are in the downward direction. The direction is negative (w) (z direction). The velocity at any point y along the bound vortex induced by the trailing semi-infinite vortices is given by $V = \frac{\Gamma}{4\pi h}$.

at $y=\frac{b}{2}$ and $y=-\frac{b}{2}$ $w(y) \to \infty$, that is not realistic solution. Instead of representing the wing by a single horseshoe vortex. This can be represented by a large number of horseshoe vortices, each with different length of bound vortex, but with all the bound vortices coincident along a single line, called the lifting line.



Let us extropolate to the case where an infinite number of horseshoe vortices are superimposed along the lifting line, each with a vinishingly small strength $d\Gamma$.

Biot-Savart Law



$$dV = \frac{\Gamma}{4\pi} \cdot \frac{dL \times r}{|r|^3}$$

$$V = \frac{\Gamma}{4\pi} \int_{-\infty}^{\infty} \frac{\sin\theta}{r^2} dL$$

$$r = \frac{h}{\sin\theta}$$

$$L = \frac{h}{\tan\theta}$$

$$dL = -\frac{h}{\sin^2\theta} d\theta$$

$$V = \frac{\Gamma}{4\pi} \int_{-\infty}^{\infty} \frac{\sin\theta}{r^2} dL$$

$$V = \frac{-\Gamma}{4\pi h} \int_{\pi}^{0} \sin\theta \, d\theta$$

$$\left[V = \frac{\Gamma}{2\pi h} \right]$$ For infinite Filament For semi-Infinite is half.

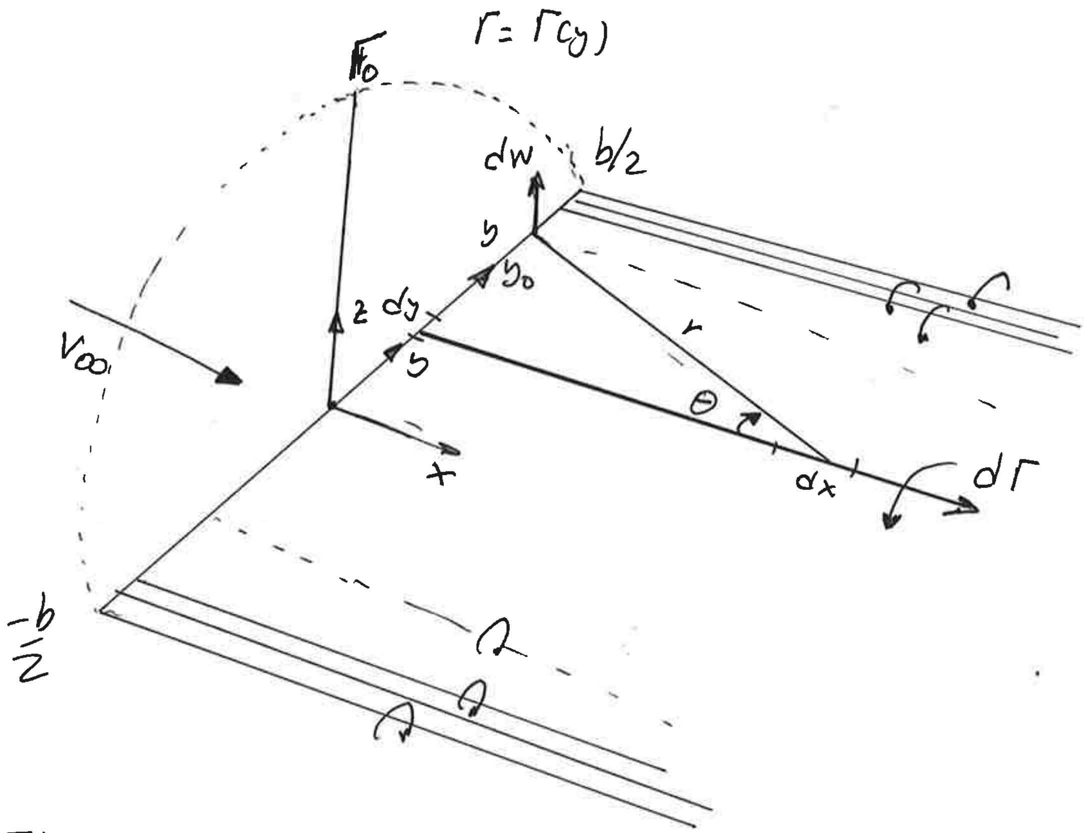
$$\left[V = \frac{\Gamma}{4\pi h} \right]$$ Semi-Infinite

Applying for our discussion. $V = w(y)$

$$w(y) = -\frac{\Gamma}{4\pi\left(\frac{b}{2} + y\right)} - \frac{\Gamma}{4\pi\left(\frac{b}{2} - y\right)}$$

$$w(y) = -\frac{\Gamma}{4\pi}\left(\frac{1}{\left(\frac{b}{2} + y\right)} + \frac{1}{\left(\frac{b}{2} - y\right)}\right)$$

$$w(y) = -\frac{\Gamma}{4\pi} \cdot \frac{b}{\left(\frac{b}{2}\right)^2 + y^2}$$



The value of the circulation at origin is Γ_0. The finite number of trailing vartéces have become a continuos vortex sheet trailing downstream of the ~~trailing~~ lifting Line. this vortex sheet is parallel to the direction of V_∞. The total strength of the sheet integrated across the span of the wing is zero, because it consists of pairs of trailing vortices of equal strength but in opposite direction.

From the figure we note that $\Gamma(y)$ function of y. then $d\Gamma = (d\Gamma/dy)\,dy$. we obtain dw as

$$dw = \frac{-\left(\frac{d\Gamma}{dy}\right) dy}{4\pi(y_0 - y)} \quad \text{by Biot-Savart}$$

The total velocity w induced at y_0 by the entire trailing vortex sheet is the summation of the above equation.

$$w(y_0) = -\frac{1}{4\pi} \int_{-b/2}^{b/2} \frac{(d\Gamma/dy).\,dy}{(y_0 - y)}$$

We want to calculate $\Gamma(y)$ for a given finite wing, along with its corresponding total lift and induced drag.

The above equation is important in that it gives the value of the downwash at y_0 due to all the trailing vortices.

Supose that local airfoil section along the wing span is at y_0.

$$\alpha_i(y_0) = \tan^{-1}\left(\frac{-w(y_0)}{V_\infty}\right)$$

As V_∞ is bigger that w, α_i is small.

For small angle. We obtain that

$$\alpha_i(y_0) = \frac{-w(y_0)}{V_\infty}$$

Then α_i is obtained as:

$$\alpha_i(y_0) = \frac{1}{4\pi V_\infty} \int_{-b/2}^{b/2} \frac{(d\Gamma/dy)\,dy}{y_0 - y}$$

That is, an expression for induced angle of attack in terms of circulation ($\Gamma(y)$) along the wing.

Consider again the effective angle of attack α_{eff}., this angle of attack is seen by the local airfoil section. As α_i varies across the span, then α_{eff} is also variable, thus $\alpha_{eff} = \alpha_{eff}(y_0)$.

The lift coefficient for the airfoil section located at $y = y_0$ is

$$C_L = \alpha_0 \left[\alpha_{eff}(y_0) - \alpha_{L=0}\right] = 2\pi \left[\alpha_{eff}(y_0) - \alpha_{L=0}\right]$$

The local section lift slope α_0 has been replaced by the thin airfoil theoritical value of 2π (rad^{-1}). Also for the wing with aerodynamic twist, the angle of zero-lift $\alpha_{L=0}$ is constant across the span.

If there is no aerodynamic twist $\alpha_{L=0}$ is constant across the span. $\alpha_{L=0}$ is a known property of the local airfoil section. From definition the local airfoil section lift at y_0 is.

$$L = \rho_\infty V_\infty \Gamma(y_0) = \frac{1}{2} \rho_\infty V_\infty^2 \, c(y_0) . C_L$$

$$C_L = \frac{2\Gamma(y_0)}{V_\infty \, c(y_0)} \quad \text{and} \quad C_L = 2\pi[\alpha_{eff}(y_0) - \alpha_{L=0}]$$

From these equations we obtain α_{eff}

$$\frac{2\Gamma(y_0)}{V_\infty \, c(y_0)} = 2\pi[\alpha_{eff}(y_0) - \alpha_{L=0}]$$

$$\left[\alpha_{eff} = \frac{\Gamma(y_0)}{\pi V_\infty \, c(y_0)} + \alpha_{L=0} \right]$$

and $\alpha_{eff} = \alpha - \alpha_i$

$$\alpha(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty \, c(y_0)} + \alpha_{L=0}(y_0) + \frac{1}{4\pi V_\infty} \int_{-b/2}^{b/2} \frac{(d\Gamma/dy)\,dy}{y_0 - y}$$

The fundamental equation of prandtl's lifting-line theory.

This states that the geometric angle of attack is equal to the sum of the effective angle plus the induced angle of attack. α_{eff} and α_i are expressed by Γ and $d\Gamma/dy$ respectively. Hence, the only unknown is Γ; all the other quantities α, c, V_∞ and $\alpha_{L=0}$ are known for a finite wing of given design at a given geometric angle of attack in a freestream with given velocity. So a solution of the above equation yields $\Gamma = \Gamma(y_0)$ and $-\frac{b}{2} < y_0 < b/2$

The solution $\Gamma = \Gamma(y_0)$ obtained gives the threee main aerodynamic characteristics of finite wing.

1.) Lift distribution is obtained from Kutta-Joukowski

$$L(y_0) = \rho_\infty V_\infty \Gamma(y_0)$$

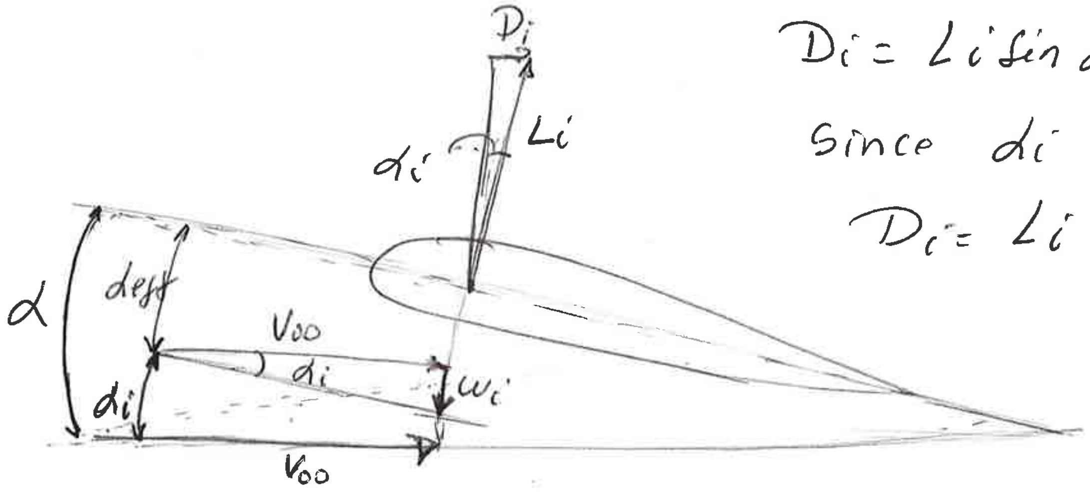
2.) The total lift is obtained by integration of the above equation over span.

$$L = \int_{-b/2}^{b/2} L(y)\,dy \quad \text{or} \quad L = \rho_\infty V_\infty \int_{-b/2}^{b/2} \Gamma(y)\,dy$$

The lift coefficient follows immediately from the equation

$$C_L = \frac{L}{\rho_\infty S} = \frac{2}{V_\infty \cdot S} \int_{-b/2}^{b/2} \Gamma(y)\,dy$$

3). The induced drag is obtained by inspection of the figure.



$D_i = L_i \sin \alpha_i$

Since α_i is small

$D_i = L_i \alpha_i$

$$D_i = \int_{-b/2}^{b/2} L(y) \cdot \alpha_i(y)\,dy$$

$$D_i = \rho_\infty V_\infty \int_{-b/2}^{b/2} \Gamma(y)\,\alpha_i(y)\,dy$$

$$C_{Di} = \frac{D_i}{\rho_\infty \cdot S} = \frac{2}{V_\infty S} \int_{-b/2}^{b/2} \Gamma(y) \cdot \alpha_i(y)\,dy$$

Elliptical Lift Distribution

A very important special case is the elliptical circulation distribution, this distribution represents the wing of minimum induced drag. Fortunaly, the properties of wings of arbitrary plantforms that do no differ radically from the usual shapes are close to chose to those of the elliptical wing.

Considor a circulation distribution given by

$$\Gamma = \Gamma_0 \cdot \sqrt{1-\left(\frac{2y}{b}\right)^2}$$

Γ_0 : circulation at the origin

The circulation varies elliptically with distance y along the span; For that, it is designated as an elliptical circulation distribution.

$$L(y) = \rho_\infty V_\infty \Gamma_0 \sqrt{1-\left(\frac{2y}{b}\right)^2}$$

Elliptical lift distribution.

$\Gamma(b/2) = \Gamma(-b/2) = 0$. So, the circulation, hence, Lift goes from zero at wing tip.

First, let's calculate the downwash.

$$w(y_0) = \frac{\Gamma_0}{\pi b^2} \int_{-b/2}^{b/2} \frac{y}{(1 - 4y^2/b^2)^{1/2} (y_0 - y)} dy$$

$$\frac{d\Gamma}{dy} = -\frac{4\Gamma_0}{b^2} \frac{y}{\left(1 - \frac{4y^2}{b^2}\right)^{1/2}}$$

We substitute $y = \frac{b}{2} \cos\theta$ and $dy = -\frac{b}{2} \sin\theta \, d\theta$.

$$w(\theta_0) = \frac{-\Gamma_0}{2\pi b} \int_{\pi}^{0} \frac{\cos\theta}{\cos\theta - \cos\theta_0} d\theta$$

$$w(\theta_0) = -\frac{\Gamma_0}{2\pi b} \int_{0}^{\pi} \frac{\cos\theta}{\cos\theta_0 - \cos\theta} d\theta$$

$$\left[w(\theta_0) = \frac{-\Gamma_0}{2b} \right]$$

This solution states that the downwash is constant over the span for an elliptical lift distribution.

And we calculate α_i based on α_i is small

$$\alpha_i = -\frac{w}{V_\infty} = -\left(\frac{-\Gamma_0}{2b}\right)\frac{1}{V_\infty} = \frac{\Gamma_0}{2bV_\infty}$$

Indicates that α_i at any point along the lifting line is constant if the Γ distribution is elliptical.

$$L(y) = \rho_\infty V_\infty \Gamma_0 \int_{-b/2}^{b/2} \left(1 - \frac{4y^2}{b^2}\right)^{1/2} dy$$

$$L(y) = \rho_\infty V_\infty \Gamma_0 \cdot \frac{b}{2} \int_0^{\pi} \sin^2\theta \, d\theta$$

$$L(y) = \rho_\infty V_\infty \Gamma_0 \frac{b}{4}\pi$$

$$\Gamma_0 = \frac{4L}{\rho_\infty V_\infty b \pi} \quad \text{and} \quad L = \frac{1}{2}\rho_\infty V_\infty^2 S . C_L$$

$$\Gamma_0 = \frac{4 . \rho_\infty V_\infty^2 . S . C_L}{2 \rho_\infty V_\infty b . \pi} = \frac{2 . V_\infty S C_L}{b\pi}$$

Then $$\alpha_i = \frac{\Gamma_0}{2bV_\infty} = \frac{2 . V_\infty S . C_L}{b . \pi . 2bV_\infty} = \frac{S . C_L}{b^2 \pi}$$

One important geometric property of a finite wing is the aspect ratio,

$$AR = \frac{b^2}{S}$$

Hence, the above equation (induced angle of attack) is given by

$$\alpha_i = \frac{C_L}{\pi AR}$$

The induced drag coefficient is obtained from

$$C_{Di} = \frac{2\alpha_i}{V_\infty S}\int_{-b/2}^{b/2} \Gamma(y)\,dy = \frac{2\alpha_i \Gamma_0}{V_\infty . S} . \frac{b}{2}\int_0^{\pi} \sin^2\theta \, d\theta$$

$$C_{Di} = \frac{\pi}{2} . \frac{\alpha_i \Gamma_0 b}{V_\infty S}$$

$$C_{Di} = \frac{\pi b}{\cancel{2}\cancel{V_\infty}\cancel{S}} * \left(\frac{C_L}{\pi AR}\right) * \frac{\cancel{2}\cancel{V_\infty^2}\cancel{S} C_L \cancel{\rho_\infty}}{\cancel{\rho_\infty}\cancel{V_\infty}\cancel{b\pi}}$$

$$\left[C_{Di} = \frac{C_L^2}{\pi AR} \right]$$

The Drag coefficient (induced) is directly proportional to the square of the lift coefficient. Also called the drag due to lift. The induced drag increases rapidly as C_L increases and becomes a substantial part of the total drag when C_L is high.

Ex.

An untwisted wing with an elliptical plantform and elliptical lift distribution has an aspect ratio of 6 and a span of 12m. The wing loading is 900 N/m2 when flying at a speed of 150 Km/h (41.67 m/s) at sea level. We shall compute the induced drag for this wing.

The projected area and the total lift of the wing are, respectively.

$$S = \frac{b^2}{AR} = 24 m^2 \qquad L = 21600\,N = 900 \times 24 \frac{N}{m^2} \cdot m^2$$

the dynamic pressure is

$$q_\infty = \frac{1}{2}(1.226)\frac{Kg}{m^3} \times (41.67)^2 \frac{m^2}{s^2} =$$

$$q_\infty = 1064\ N/m^2$$

$$C_L = C_L = \frac{L}{q_\infty S} = 0.846$$

$$C_{di} = C_{Di} = \frac{C_L^2}{\pi AR} = 0.038$$

the induced drag is

$$D_i = L \frac{C_{Di}}{C_L} = 970\,N$$

which is 4.49% of the total lift.

the additional power that is required to compensate for the induced drag of this finite wing

$$P = D_i V_\infty = 970.\,N.\ 41.67\,\frac{m}{s} = 54.2\,hp.$$

The induced angle of attack and the constant downwash.

$$\alpha_i = -\frac{C_{Di}}{C_L} = -0.045\,rad.\ (-2.58°)$$

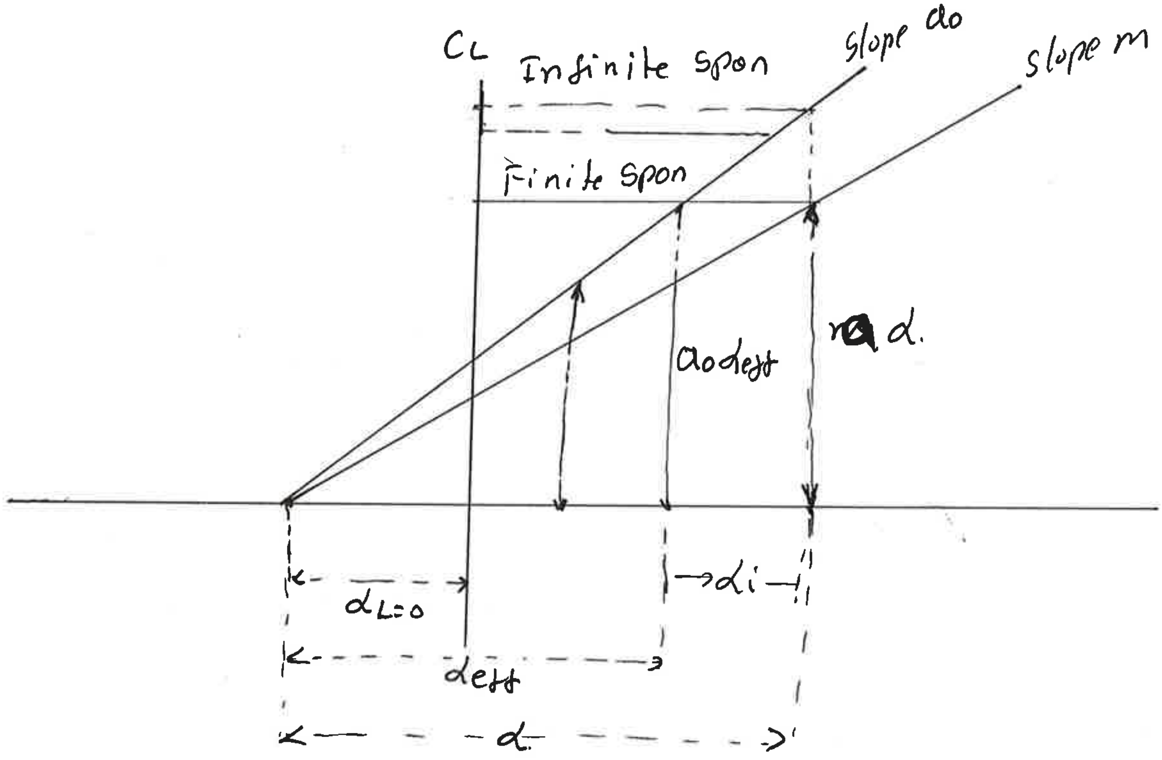
$$w = -\frac{\alpha_i}{\cancel{V_\infty}} \cdot V_\infty = 0.045 \times 41.67 = 1.88\,m/s$$

The effective angle of attack and the absolute angle of attack are constant.

$$\alpha_{eff} = \alpha - \alpha_i \Rightarrow \alpha_{eff} = \frac{C_L}{2\pi} \quad \text{(Thin theory airfoil)}$$

$$\alpha_{eff} = \frac{0.846}{2\pi} = 0.135\,(rad)\ (7.73°)$$

$$\alpha = \alpha_{eff} + \alpha_i = (0.135 + 0.045) = 0.18\,rad.\ (10.31°)$$



$$\alpha = \alpha_{eff} + \alpha_i$$

$$\frac{C_L}{a} = \frac{C_L}{a_0} + \alpha_i$$

$$|a| = \frac{a_0}{1 + \frac{\alpha_i . a_0}{C_L}} = \frac{a_0}{1 + \frac{\alpha_i}{\alpha_{eff}}}$$

$$\left[a = \frac{a_0}{1 + \frac{0.045}{0.135}} = 0.75\, a_0 \right]$$

This shows that the finite wing, in this example, generates only 75% of the Lift that would be generated by the same wing if the effect of induced downwash were ignored

The integral is $\int_0^{\pi} \sin(n\theta).\sin\theta\, d\theta = \begin{cases} \pi/2 & \text{for } n=1 \\ 0 & \text{for } n\neq 1 \end{cases}$

$C_L = A_1.\pi.\frac{b^2}{S} = A_1.\pi.AR$ (only depends on the leading coefficient of the Fourier series expansion).

Then we must solve for all the A_n's simultaneously in order to obtain A_1.

The induced drag coefficient is obtained from

$$C_{Di} = \frac{2}{V_\infty S}\int_{-b/2}^{b/2} \Gamma(y).\alpha_i(y)\,dy$$

$$= \frac{2b^2}{S}\int_0^{\pi}\left(\sum_1^N A_n.\sin(n\theta)\right).\alpha_i(\theta).\sin\theta\, d\theta$$

where $\alpha_i(\theta)$ is obtained by

$$\alpha_i(y_0) = \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2}\frac{(d\Gamma/dy).dy}{y_0-y}$$

$$\alpha_i(\theta_0) = \frac{1}{\pi}.\sum_1^N n.A_n.\int_0^{\pi}\frac{\cos(n\theta)}{\cos\theta-\cos\theta_0}\,d\theta$$

$$\alpha_i(\theta_0) = \sum_1^N n.A_n.\frac{\sin(n\theta_0)}{\sin\theta_0}$$

can be written as

$$d_i(\theta) = \sum_{1}^{N} n . A_n . \frac{\sin(n\theta)}{\sin\theta}$$

We obtain C_{Di} as

$$C_{Di} = \frac{2b^2}{S} \int_0^{\pi} \sum_{1}^{N} A_n . \sin(n\theta) . \sum_{1}^{N} A_n . n . \sin(n\theta)\, d\theta$$

From the standard integral.

$$\int_0^{\pi} \sin(n\theta) . \sin(n\theta) = \int_0^{\pi} \sin(m\theta) . \sin(k\theta)\, d\theta \begin{cases} 0 & m \neq k \\ \frac{\pi}{2} & m = k \end{cases}$$

$$C_{Di} = \frac{2b^2}{S} . \sum_{1}^{N} (n A_n^2) . \frac{\pi}{2} = 2 . AR . \sum_{1}^{N} (n A_n^2)$$

$$C_{Di} = \pi AR (A_1^2 + \sum_{2}^{N} n A_n^2)$$

$$= \pi AR . A_1^2 . \left[1 + \sum_{2}^{N} n \left(\frac{A_n}{A_1}\right)^2 \right]$$

$$C_{Di} = \frac{C_L^2}{\pi AR} (1 + \delta) \qquad \text{where } \delta = \sum_{2}^{N} n \left(\frac{A_n}{A_1}\right)^2 \geq 0$$

Let us define a span efficiency factor, e as $e = (1+\delta)^{-1}$

$$\boxed{C_{Di} = \frac{C_L^2}{\pi . e AR}}$$

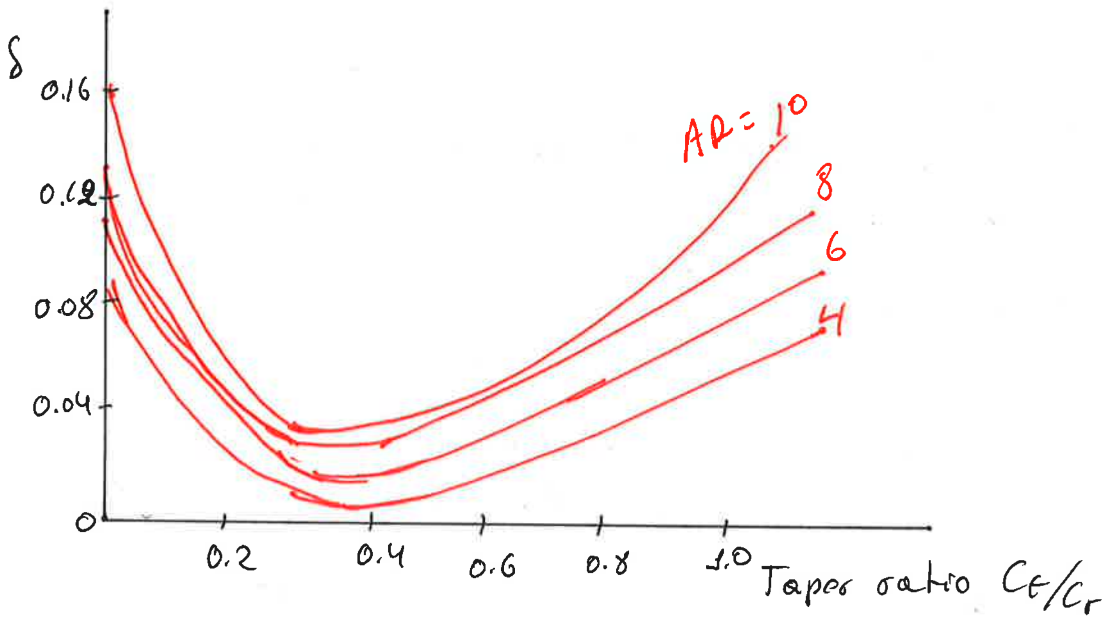
$e \leq 1$ for general lift distribution.
For elliptical lift distribution
$\delta = 0$ and $e = 1$

Hence the lift distribution which yields minimum induced drag is the elliptical lift distribution. This is why we have a practical interest in the elliptical lift distribution.

Elliptic wing: Elliptical planform, are more expensive to manufacture.

Rectangular wing: Generates a lift distribution far from optimum.

Tapered wing: Can be designed with taper ratio, that is Tip chord/root chord $\equiv C_t/C_r$, such that the lift distribution closely approximates the elliptic case.



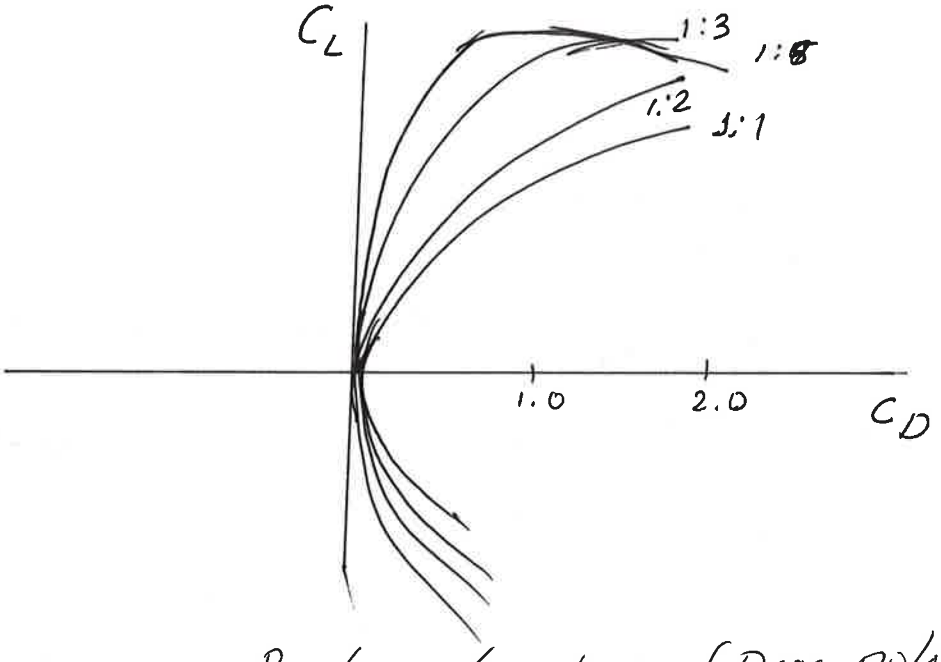
The variation of δ as function of taper ratio for wings of different A.R. With this graph a tapered wing can be designed with an induced drag coefficient reasonably close to the minimum value. The manufacture is considerable easier to manufacture than elliptic plantform.

Effect of Aspect Ratio

The AR. varies from 6 to 22 for standard subsonic airplanes and sailplanes. has much stronger effect on C_{Di} than the value of δ. which from the above graph varies only by about 10 percent over the practical range of taper ratio. So that the primary design factor for minimizing induced drag is not the closeness to an elliptical lift distribution, but rather, the ability to make the aspect ratio as larger as possible.

Recall that. the total drag of finite wing is giving by

$$C_D = C_d + \frac{C_L^2}{\pi e AR}$$



Rectangular wing. (Drag polar)

Consider two wings with different aspect ratio AR_1 and AR_2. The Drag Coefficients C_{D1} and C_{D2} for the two wings as

$$C_{D_1} = C_d + \frac{C_L^2}{\pi e AR_1}$$

$$C_{D2} = C_d + \frac{C_L^2}{\pi e . AR_2}$$

Assume that the wings are at the same C_L. Also since the airfoil section is the same for both wings. C_d. is essencially the Same.

Also the varration of e between the wing is only a few percent and can be ignored.

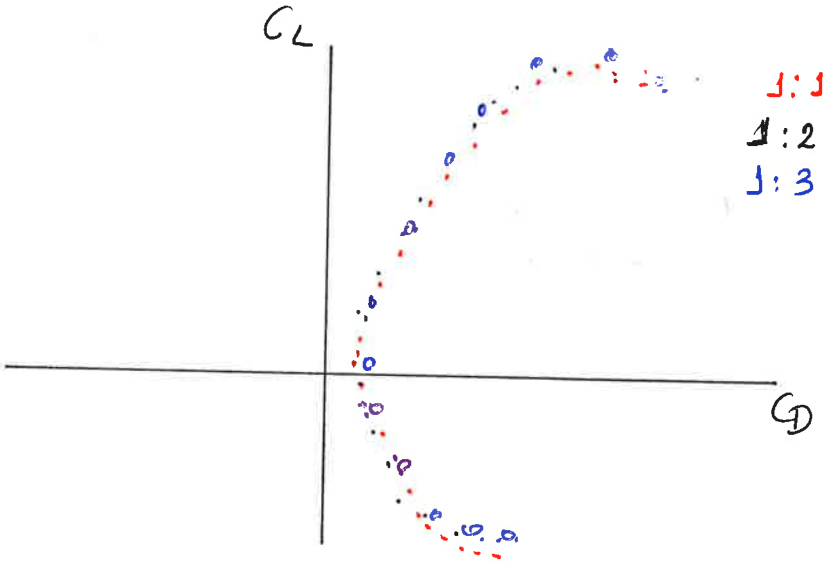
$$C_{D_1} = C_{D_2} + \frac{C_L^2}{\pi e}\left(\frac{1}{AR_1} - \frac{1}{AR_2}\right)$$

This equation can be used to scale the data of a wing with aspect ratio AR_2 to correspond to the case of another aspect ratio AR_1.

Ex.

$$C_{D1} = C_{D2} + \frac{C_L^2}{\pi e}\left(\frac{1}{5} - \frac{1}{AR_2}\right)$$

Inserting the value of C_{D2} and AR_2 from the above graph. the resulting data for C_{D1} vs. C_L collapsed to essentially the same curve.



Airfoil and finite wing have some differences. Finite wing produces a Induced drag. Second major difference appears in the lift slope.

The slope of airfoil $a_0 = \frac{dC_L}{d\alpha}$

The slope of wing $a = \frac{dC_L}{d\alpha}$

where $a < a_0$. the values of a_0 and a are related as follows.

$$\frac{dC_L}{d(\alpha - \alpha_i)} = a_0$$

$$C_L = a_0(\alpha - \alpha_i) + \text{constant}$$

$\alpha_i = \frac{C_L}{\pi AR}$ into

$$C_L = a_0(\alpha - \alpha_i) + \text{constant}$$

$$C_L = a_0\left(\alpha - \frac{C_L}{\pi AR}\right) + \text{constant}$$

Differentiating.

$$= \frac{dC_L}{d\alpha} = a = \frac{a_0}{1 + a_0/\pi AR}$$

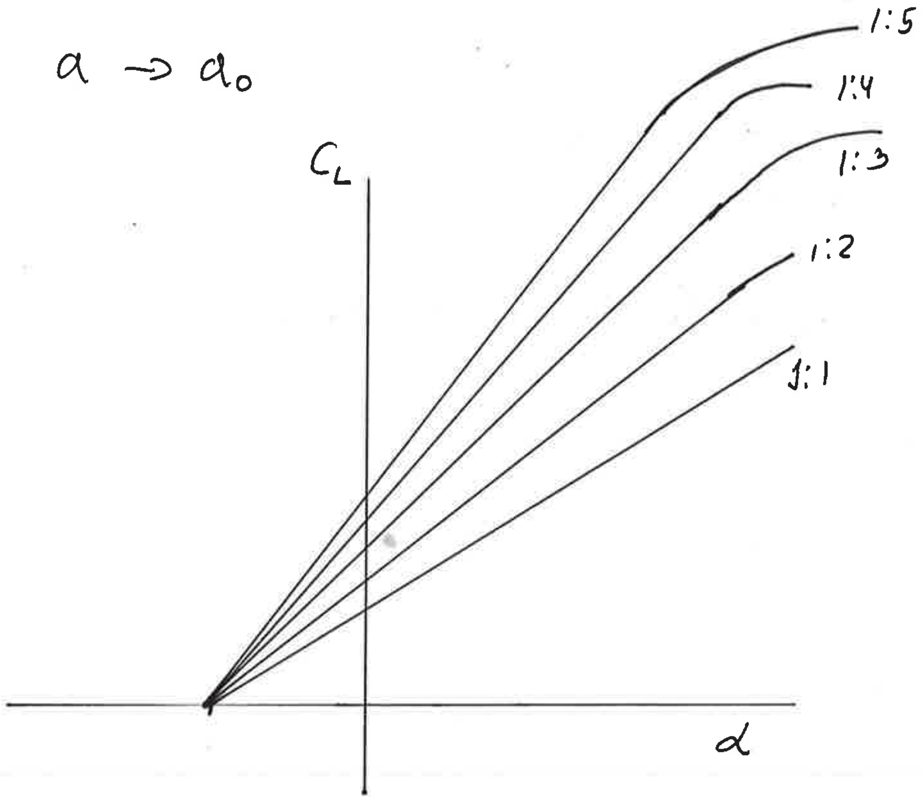
This equation gives the desired relation between a_0 and a for an elliptic finite wing.

$$a = \frac{a_0}{1 + (a_0/\pi AR)(1+\tau)}$$

This equation is for general planform where τ is a function of Fourier Coefficients A_n. The values of τ range between 0.05 and 0.25.

$AR \to 0$, a substantial difference can exit between a_0 and a.

$AR \to \infty$ $\quad a \to a_0$



Example

The analysis of this section is now applied to compute the characteristics of an untwisted rectangular wing $AR = 6$ flying with α. Assume that the airfoil is uncambered so that the absolute of attack α_a is equal to α everywhere along the span.

$C = C_s$ and $a = a_0 = 2\pi$, the equation of α is reduced to

$$\alpha = \sum_{n=1}^{\infty} A_n \sin(n\theta)\left(1 + \frac{n\pi}{2AR\sin\theta}\right)$$

For symmetrically loaded wing, the coefficients A_n vanish for even values of n.

$$\alpha = A_1 \sin\theta\left(1 + \frac{\pi}{6\times 2.\sin\theta}\right) + A_3 .\sin 3\theta\left(1 + \frac{\pi}{4\sin\theta}\right)$$

$$+ A_5 \sin(5\theta).\left(1 + \frac{5\pi}{12\sin\theta}\right) + A_7 \sin 7\theta\left(1 + \frac{7\pi}{12.\sin\theta}\right)$$

We take only one-half of the span because of the symmetry of the rectangular wing. so we take only the four stations of the wing.

$\theta = \frac{\pi}{8}$, $\theta = \frac{\pi}{4}$; $\theta = \frac{3\pi}{8}$ and $\theta = \frac{\pi}{2}$

We obtain a set of equations.

$$0.6444 A_1 + 2.8200 A_3 + 4.0841 A_5 + 2.2153 A_7 = \alpha$$

$$0.9689 A_1 + 1.4925 A_3 + 2.0161 A_5 + 2.5397 A_7 = \alpha$$

$$1.1857 A_1 + 0.7080 A_3 - 0.9244 A_5 + 2.7565 A_7 = \alpha$$

$$1.261 A_1 - 1.7854 A_3 + 2.3090 A_5 - 2.8326 A_7 = \alpha$$

$A_1 = 0.9174\alpha$ $A_3 = 0.1104\alpha$ $A_5 = 0.0218\alpha$ $A_7 = 0.0038\alpha$

The wing-lift coefficient is

$$C_L = \pi^2 \frac{A_1}{2} = 4.5273\alpha$$

Based on the values.

$$(C_{Di})_{el} = \frac{C_L^2}{\pi AR} = 1.0874\alpha^2$$

and $$\delta = \frac{3 A_3^2 + 5 A_5^2 + 7 A_7^2}{A_1^2} = 0.0464$$

Induced Drag for the wing.

$$C_{Di} = C_{Di\,el}(1+\delta) = 1.1378\alpha^2$$ this higher about 5% of $(C_{Di})_{el}$

Calculating the lift coefficient

$$(C_L)_{el} = \frac{2\pi \cdot \alpha}{1 + \frac{2\pi}{\pi AR}} = 1.5\pi\alpha = 4.7124\alpha.$$

So the slope of the lift coefficient curves, $dC_L/d\alpha$ of rectangular wing is approximately 4% lower.

The result verifies the properties of arbitrary planforms are close to those of elliptical wing.

Ex. 2

Consider a finite wing with an aspect ratio of 8 and taper ratio of 0.8. The airfoil section is thin and symmetric. Calculate the lift and induced drag coefficients for the wing when it is an angle of attack of 5°. Assume that $\delta = \tau$

Solution

from graph $\delta = 0.055$ and $\tau = \delta = 0.055$ and $a_0 = 2\pi$.

$$C_L = \alpha a \quad \Rightarrow \quad a = \frac{a_0}{1 + a_0/\pi AR(1+\tau)}$$

$C_L = 4.97 \text{ rad}^{-1}$

$C_L = 0.0867 \text{ Degree}^{-1}$

Since airfoil is symmetric $\alpha_L = 0 = 0^\circ$

Then $C_L = a.\alpha = (0.0867.. \times 5 \text{ Degree}) = 0.4335$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ Degree

$$C_{D_i} = \frac{C_L^2}{\pi AR e} = 0.00789 \qquad e = \frac{1}{1+\delta}$$

Ex. Jet transport patterned after Cesna 560 citation V. shown a drag coefficient of cruise 0.015. AnAddition the zero-lift angle of attack is -2°, the lift slope of the airfoil section is 0.1 per degree. The lift efficienty factor $\tau = 0.04$ and the wing aspect ratio is 7.96. Calculate the angle of attack of the airplane.

Solution.

The lift. slope of the airfoil is in radian

$$a_0 = 0.1 \text{ Prer dgree} = 0.1 \times (57.3) = 5.73 \text{rad.}$$

$$a = \frac{a_0}{1 + (a_0/\pi AR)(1+\tau)}$$

$$a = \frac{5.73}{1 + \left(\frac{5.73}{\pi .\ 7.96}\right)(1+0.04)} = 4.627 \text{ per rad.}$$

$$a = \frac{4.627}{57.3} = 0.0808 \text{ per degree}$$

$$C_L = a(\alpha - \alpha_{L=0})$$

Cruising Data is : $V = 790\ Km/h$ $\rho = 0.41\ kg/m^3$

$W = 68KN$ plan form area $= 31.8 m^2$

Then $C_L = \dfrac{2W}{\rho_\infty V_\infty^2 S} = \dfrac{2 \cdot 68000}{(0.41)\cdot(219.4)^2\cdot(31.8)} = 0.21$

$$\alpha = \frac{C_L}{a} + \alpha_{L=0} = \frac{0.21}{0.0808} + (-2) = 0.6^\circ$$

Ex. Consider a rectangular wing on aspect ratio of 6. an induced drag factor $\delta = 0.055$ and $\alpha_{L=0} = -2^\circ$. and angle of attack of 3.4°. The induced drag coefficient for this wing is 0.01. Calculate the induced drag coefficient for a similar wing. at the same angle of attack but with an AR. 10. Induced factors for drag and lift slope δ and τ are equal to each other AR 10, $\delta = 0.105$

Solution

$$C_{Di} = \frac{C_L^2}{\pi AR e}$$ we need to determine C_L

For AR. 6 the C_L is

$$C_L^2 = \frac{\pi AR. C_{Di}}{(1+\delta)} = \frac{\pi. 6 \times (0.01)}{(1 + 0.055)} = 0.1787$$

$$C_L = 0.423$$

The lift slope of this wing is therefore

$$\frac{dC_L}{d\alpha} = \frac{0.423}{(3.4° - (-2°))} = 0.078/\text{degree} = 4.485/\text{rad}$$

The lift slope for the airfoil (the infinite wing).

$$\frac{dC_L}{d\alpha} = a = \frac{a_0}{1 + (a_0/\pi AR)(1+\tau)}$$

$$4.485 = \frac{a_0}{1 + \left(\frac{(1.055).\, a_0}{\pi. 6}\right)} = \frac{a_0}{1 + 0.056 a_0}$$

$$a_0 = \frac{5.989}{\text{rad}}$$

The second wing has the same airfoil section

Then a_0 is the same.

The lift slope for the second wing.

$$a = \frac{a_0}{1 + \left(\frac{(1.105 \times a_0)}{\pi. 10}\right)} = 4.95/\text{rad}$$

$$a = 0.086/\text{degree.}$$

The lift coefficient for the second wing is therefore

$$C_L = a(\alpha - \alpha_{L=0}) = 0.086\,[3.4^\circ - (-2^\circ)$$

$$C_L = 0.464$$

In turn, the induced drag coefficient is

$$C_{Di} = \frac{C_L^2}{\pi AR}(1+\delta) = \frac{0.464\,(1+0.105)}{\pi \cdot 10} = 0.0076$$

Consider same C_L rather than α. for both wing.

$$C_{Di} = \frac{C_L^2}{\pi AR}(1+\delta) = \frac{(0.423)^2(1+0.105)}{\pi \cdot 10} = 0.0063.$$

One observation about induced Drag D_i; and contrast to the induced drag coefficient C_{Di}, that C_{Di} decreases by increases the AR. The Drag (D_i) is governed by other parameter.

$$D_i = q_\infty S \cdot C_{Di} = q_\infty \cdot S \cdot \frac{C_L^2}{\pi e AR} \quad \text{for steady flight}$$

$$C_L^2 = \left(\frac{L}{q_\infty S}\right)^2 = \left(\frac{W}{q_\infty S}\right)^2$$

$$D_i = q_\infty \cdot S \cdot \frac{W^2}{q_\infty^2 S^2} \cdot \frac{1}{\pi e AR}$$

$$D_i = \frac{1}{\pi e}\frac{1}{q_\infty}\left(\frac{W^2}{b^2}\right)$$ For steady flight induced Drag does not depend on the aspect ratio; but rather other parameter $\left(\frac{W}{b}\right)$ called the span loading : span loading $= \frac{W}{b}$. Drag (induced) can be reduced by increasing b. span.

How much of the total drag of an airplane is induced drag?

The parasite drag for a generic subsonic jet transport is the sum of the drag due to skin friction and pressure drag due to flow separation associated with the complete airplane including wing. At cruise about 25% is induced drag but at takeoff can be 60% or more of the total drag.