Vector Spaces 4 Given an edomorphism in $\mathbb{R}^4$ that works: $\begin{cases} f(\bar{e}_1) = \bar{e}_1 + \bar{e}_2 \\ f(\bar{e}_1 + 2\bar{e}_2) = 3\bar{e}_1 + 3\bar{e}_2 \\ f(\bar{e}_3 + \bar{e}_4) = \bar{0} \\ f(\bar{e}_4) = 2\bar{e}_3 \end{cases}$

$B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4\}$

$$\boxed{f(\lambda\bar{x} + \mu\bar{y}) = \lambda f(\bar{x}) + \mu f(\bar{y})}$$

Obtain:

1. F_B ?
2. $Ker(f)$ and $Im(f)$?
3. Transform $\bar{u} = (1,1,1,1)$
4. Obtain the set of vectors that transform into the vector $\bar{w} = (1,1,0,0)$

1.

$$\begin{cases} f(\bar{e}_1) = \bar{e}_1 + \bar{e}_2 \longrightarrow f(\bar{e}_1) = (1,1,0,0)_B \\ f(\bar{e}_1 + 2\bar{e}_2) = 3\bar{e}_1 + 3\bar{e}_2 \longrightarrow f(\bar{e}_1) + 2f(\bar{e}_2) = 3\bar{e}_1 + 3\bar{e}_2 \\ f(\bar{e}_3 + \bar{e}_4) = \bar{0} \longrightarrow f(\bar{e}_3) + f(\bar{e}_4) = \bar{0} \\ f(\bar{e}_4) = 2\bar{e}_3 \longrightarrow f(\bar{e}_4) = (0,0,2,0)_B \end{cases}$$

$$F_B = \begin{array}{c} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{matrix} f(\bar{e}_1) & f(\bar{e}_2) & f(\bar{e}_3) & f(\bar{e}_4) \end{matrix} \end{array}$$

$$f(\bar{e}_1) + 2f(\bar{e}_2) = 3\bar{e}_1 + 3\bar{e}_2 \longrightarrow f(\bar{e}_2) = \frac{1}{2}\left(3\bar{e}_1 + 3\bar{e}_2 - f(\bar{e}_1)\right) = \bar{e}_1 + \bar{e}_2 = (1,1,0,0)_B$$

$$f(\bar{e}_3) + f(\bar{e}_4) = \bar{0} \longrightarrow f(\bar{e}_3) = -f(\bar{e}_4) = -2\bar{e}_3 = (0,0,-2,0)_B$$

2.

$$F_B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$f(\bar{e}_1)\ f(\bar{e}_2)\ f(\bar{e}_3)\ f(\bar{e}_4)$

$\dim(\mathrm{Im}(f)) = Rg(F_B) = 2$

$\dim(\underbrace{\cancel{\mathrm{Im}(f)}}_{2}) + \dim(\mathrm{Ker}(f)) = \dim(\underbrace{\cancel{\mathbb{R}^4}}_{4}) \longrightarrow \dim(\mathrm{Ker}(f)) = 4 - 2 = 2$

$B_{\mathrm{Im}} = \{(1,1,0,0), (0,0,1,0)\}$

$B_{\mathrm{Ker}} = \{(1,-1,0,0), (0,0,1,1)\}$

$$\boxed{\begin{array}{l} f(\bar{e}_1) - f(\bar{e}_2) = \bar{0} \rightarrow f(\bar{e}_1 - \bar{e}_2) = \bar{0} \\ \bar{e}_1 - \bar{e}_2 \in \mathrm{Ker}(f) \\ f(\bar{e}_3) + f(\bar{e}_4) = \bar{0} \rightarrow f(\bar{e}_3 + \bar{e}_4) = \bar{0} \\ \bar{e}_3 + \bar{e}_4 \in \mathrm{Ker}(f) \end{array}}$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{F_B} \underbrace{\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}}_{\bar{x}} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\bar{0}}$$

$$\left\{ \begin{array}{l} x^1 + x^2 = 0 \\ \cancel{x^1 + x^2 = 0} \\ -2x^3 + 2x^4 = 0 \\ \cancel{0 = 0} \end{array} \right. \longrightarrow \left\{ \begin{array}{l} x^1 = \alpha \\ x^2 = -\alpha \\ x^3 = \beta \\ x^4 = \beta \end{array} \right. \quad \forall \alpha, \beta \in \mathbb{R}$$

3. $f(\bar{u}) = (2,2,0,0)_B$

$$F_B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\qquad\quad f(\bar{e}_1)\ f(\bar{e}_2)\ f(\bar{e}_3)\ f(\bar{e}_4)$

$$(F_B)\underbrace{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}_B}_{\bar{u}} = \underbrace{\begin{pmatrix}2\\2\\0\\0\end{pmatrix}_B}_{f(\bar{u})}$$

This MAY happen at times:

$$\left.\begin{array}{l} \dim(Ker(f)) = 0 \\ Ker(f) = \{\bar{0}\} \end{array}\right\} \iff f \text{ is INYECTIVE}$$

ALSO this has NOTHING to do with the current exercise. It just came up in class and I thought it was a good idea...

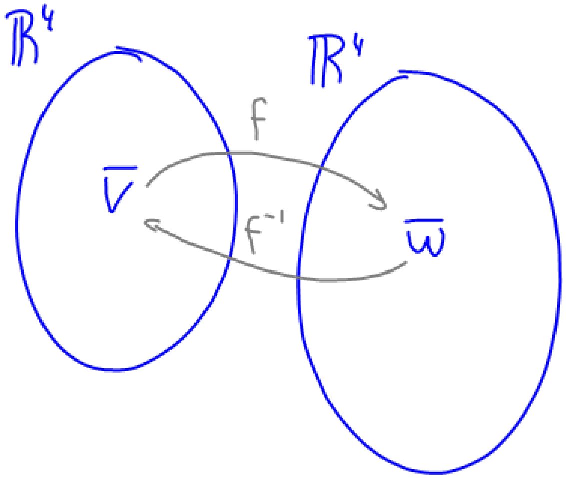
4.

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{F_B} \underbrace{\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}}_{\bar{x}} = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{\bar{w}}$$

$$f^{-1}(\bar{w}) = \{\bar{x} \in \mathbb{R}^4 / f(\bar{x}) = \bar{w}\}$$

$$\begin{cases} x^1 + x^2 = 1 \longrightarrow x^1 = 1 - x^2 \\ \cancel{x^1 + x^2 = 1} \\ -2x^3 + 2x^4 = 0 \longrightarrow x^3 = x^4 \\ \cancel{0 = 0} \end{cases} \qquad \begin{cases} x^1 = 1 - \alpha \\ x^2 = \alpha \\ x^3 = \beta \\ x^4 = \beta \end{cases} \forall \alpha, \beta \in \mathbb{R}$$

$$f^{-1}(\bar{w}) = \{\bar{x} \in \mathbb{R}^4 / (1-\alpha, \alpha, \beta, \beta)\ \forall \alpha, \beta \in \mathbb{R}\}$$



If $f(\bar{v}) = \bar{w} \longrightarrow \bar{v} \in f^{-1}(\bar{w})$

$\forall\ \bar{a} \in Ker(f)$

Then: $f(\bar{v} + \bar{a}) = f(\bar{v}) + f(\bar{a}) = \bar{w} + \bar{0} = \bar{w}$

So: $\bar{v} + \bar{a} \in f^{-1}(\bar{w})$

$$f^{-1}(\bar{y}) = \underbrace{\underbrace{\bar{x}}_{\text{Particular Result}} + \overbrace{Ker(f)}^{\text{Homgeneous Result}}}_{\text{General Result}}$$

Alternate form of 4.

$$\underbrace{f(\bar{e}_1) = \underbrace{(1,1,0,0)}_{\bar{w}}}_{\text{We see this in the exercise data}} \longrightarrow f^{-1}(\bar{w}) = \bar{e}_1 + Ker(f) = \left\{ \underbrace{(1,0,0,0)}_{\bar{e}_1} + \underbrace{\alpha(1,-1,0,0) + \beta(0,0,1,1) \ \forall \alpha,\beta \in \mathbb{R}}_{Ker(f)} \right\}$$

$$\begin{cases} f(\bar{e}_1) = \bar{e}_1 + \bar{e}_2 \\ f(\bar{e}_1 + 2\bar{e}_2) = 3\bar{e}_1 + 3\bar{e}_2 \\ f(\bar{e}_3 + \bar{e}_4) = \bar{0} \\ f(\bar{e}_4) = 2\bar{e}_3 \end{cases}$$

$$f^{-1}(\bar{w}) = \left\{ (1+\alpha, -\alpha, \beta, \beta) \ \forall \alpha, \beta \in \mathbb{R} \right\}$$

Vector Spaces 5

Space: $\mathbb{R}^3(\mathbb{R})$

$B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$

Given 3 endomorphisms in $\mathbb{R}^3$: f, g and h defined by:

$$F_B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \begin{cases} g(\bar{e}_1) = \bar{0} \\ g(\bar{e}_2) = \bar{e}_1 + \bar{e}_2 \\ g(\bar{e}_3) = \bar{e}_3 \end{cases} \qquad h = \underbrace{f \circ g}_{\text{g composed with f}}$$

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Obtain Ker and Im for each.

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$F_B = \begin{array}{c} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ f(\bar{e}_1)\ f(\bar{e}_2)\ f(\bar{e}_3) \end{array}$$

$$\dim(\mathrm{Im}(f)) = Rg(F_B) = 3$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -1 \neq 0 \rightarrow Rg(F_B) = 3$$

$$\dim(\mathrm{Ker}(f)) = \underbrace{\dim(\mathbb{R}^3)}_{3} - \underbrace{\dim(\mathrm{Im}(f))}_{3} = 0$$

$$\mathrm{Ker}(f) = \{\bar{0}\}$$

$$\dim(\mathrm{Im}(f)) = \dim(\mathbb{R}^3) \longrightarrow \mathrm{Im}(f) = \mathbb{R}^3$$

$$B_{\mathrm{Im}} = B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$$

2. $g: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$\begin{cases} g(\bar{e}_1) = \bar{0} \longrightarrow \bar{e}_1 \in Ker(g) \\ g(\bar{e}_2) = \bar{e}_1 + \bar{e}_2 \\ g(\bar{e}_3) = \bar{e}_3 \end{cases}$$

$$G_B = \begin{array}{ccc} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ g(\bar{e}_1)\ g(\bar{e}_2)\ g(\bar{e}_3) \end{array}$$

$dim(Im(g)) = R_g(G_B) = 2 \qquad B_{Im} = \{(1,1,0),(0,0,1)\}$

$dim(Ker(g)) = 3 - 2 = 1 \qquad B_{Ker} = \{\underbrace{(1,0,0)}_{\bar{e}_1 \in Ker(g)}\}$

3. $h = f \circ g$

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "g"] \arrow[rr, bend right, "f\circ g"'] & \mathbb{R}^3 \arrow[r, "f"] & \mathbb{R}^3 \\[-1.5em] \bar{x} & \bar{y} & \bar{z} \end{tikzcd}$$

$$\left.\begin{array}{l} g(\bar{x}) = \bar{y} \\ f(\bar{y}) = \bar{z} \end{array}\right\} f\circ g(\bar{x}) = \bar{z}$$

$$H_B = F_B \cdot G_B = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{F_B} \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{G_B} = \underbrace{\begin{array}{ccc} h(\bar{e}_1) & h(\bar{e}_2) & h(\bar{e}_3) \\ \multicolumn{3}{c}{\begin{pmatrix} 0 & 2 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}} \end{array}}_{H_B}$$

$R_g(H_B) = 2 = dim(Im(h)) \longrightarrow B_{Im} = \{(2,2,1),(1,0,0)\}$

$dim(Ker(h)) = dim(\mathbb{R}^3) - dim(Im(h)) = 3 - 2 = 1$

$B_{Ker} = \{(1,0,0)\}$