TEMA 6: CONTRASTE DE HIPÓTESIS

10-05-2017

X v.a. $\{F_\theta : \theta \in \Theta\}$

PASO 1 → establecer hipótesis

$H_0 : \theta \in \Theta_0$ — $\Theta = \Theta_0 \cup \Theta_1$

$H_1 : \theta \in \Theta_1$ — $\Theta_0 \cap \Theta_1 = \emptyset$

Exhaustivas excluyentes

H_0 → Hipótesis nula

H_1 → Hipótesis alternativa

$H_0 : \theta = \theta_0$ hipótesis simple

$\theta \geq \theta_0$

$\theta \leq \theta_0$

$\theta \neq \theta_0$

Hipótesis compuestas

$H_0 : \theta = \theta_0$ | $H_0 : \theta = \theta_0$

$H_1 : \theta = \theta_1$ | $H_1 \; \theta \neq \theta$

2 COLAS

$H_0 : \theta \leq \theta_0$ | $H_0 : \theta \geq \theta_0$

$H_1 : \theta > \theta_0$ | $H_1 \; \theta < \theta_1$

H compuesta / compuesta una cola

PASO 2.1

Elegir el estadístico que resuma adecuadamente la información

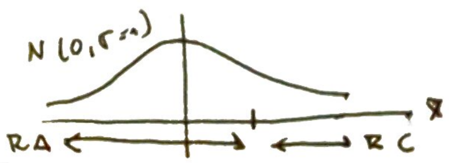
PASO 2.2.

Se divide el espacio muestral en dos regiones

→ R.A. Región de aceptación

→ R.C. Región crítica o región de rechazo

→ **Def**: Región crítica: Región del espacio muestral $\mathcal{X}$ tal que si el punto muestral $\vec{x} = (x_1, \dots, x_n) \in R.C.$ rechazaremos H_0 (la hipótesis nula) en favor de H_1 (hip. alternativa



PASO 2.3.

Se toma una m.a.s. de tamaño n, se calcula el valor del estadístico para muestra

PASO 2.4.

Se resuelve el contraste.

→ Si el valor númerico cae en la región crítica, rechazamos Ho en favor de H_1.

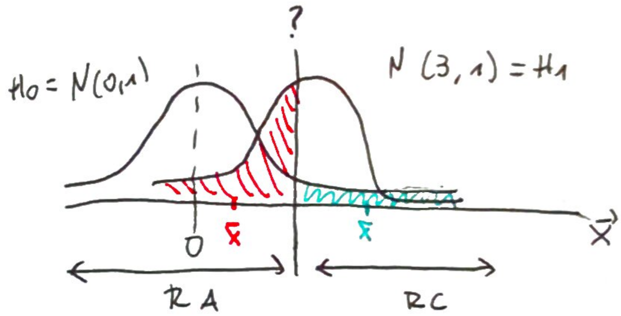
→ Si el valor númerico cae en la RA no se rechaza, no tengo evidencia para rechazar la hipótesis nula.

Ejemplo

$X \sim N(\theta, 1)$

$\begin{cases} H_0 : \theta = 0 \\ H_1 : \theta = 3 \end{cases}$

$T(\vec{x}) = \vec{x}$



$\mathcal{X} = R.A. \cdot RC$

$\bar{x}$, $T(\vec{x}) = \vec{x}$ TEST. DE LA REGIÓN CRÍTICA

$\vec{x} \in RA$ H_0 ✓

$\vec{x} \in RC$ H_1 ✓

→ Puede ser que lo cierto		pro	error
H_0	$\theta \in \Theta_0$	• $\vec{x} \in RC$	TIPO I ← Rechazo hip nula siendo
H_1	$\theta \in \Theta_1$	• $\vec{x} \in RA$	TIPO II ← Aceptar hip. nula siendo

→ P_θ (Error Tipo I) = Más grave Tipo I que Tipo II

P_θ (Rechazar H_0 | $\underbrace{\text{Ho verdadera}}_{\theta \in \Theta_0}$) ← Inocente en la carcel

→ P_θ (Error Tipo II) =

P_θ (Aceptar H_0 | $\underbrace{\text{Ho falsa}}_{\theta \in \Theta_1}$) ← culpable en la calle

FUNCIÓN POTENCIA DE UN TEST

Tamaño test
$\sup_{\theta \in \Theta_0} P(Rc)$

$\beta(\theta) = P_\theta(R.c) \qquad \theta \in \Theta = \Theta_0 \cup \Theta_1$

→ Si $\theta \in \Theta_0$, $\beta(\theta) = P_\theta(Rc) = P(\text{Error tipo I})$

Quiero que $\beta(\theta)$ sea pequeña cuando $\theta \in \Theta_0$ ← Priorizo este error

→ Si $\theta \in \Theta_1$, $\beta(\theta) = 1 - P_\theta(\text{ERROR Tipo II})$

Quiero que $\beta(\theta)$ sea máximo para que $P_\theta(ETII)$ sea mínima

Ejercicio

$N(\mu, \sigma^2 = 125)$

m.a.s tamaño $n = 25$

$\begin{cases} H_0\ \mu = 60 & \theta \in \Theta_0 \\ H_1\ \mu = 65 & \theta \in \Theta_1 \end{cases}$

Test: R.C.

$Rc = \{\bar{x} : \bar{X} \geq \lambda\}$

$\lambda = 61{,}437$

Calcula la función potencia del test

$Rc = \{\bar{x} \geq 61{,}437\}$

$\beta(\mu) = P_\mu(\bar{x} \geq 61{,}437) = 1 - P(\bar{X} \leq 61{,}437) = 1 - P\Big(\underbrace{\frac{\bar{X} - \mu}{\sigma}\sqrt{n}}_{N(0,1)} \leq \frac{61{,}437}{\sigma\sqrt{n}}\Big) =$

para $\mu = 60 ; 65$

$\bar{X} \sim N(\mu, \sigma^2/n)$ Tipifico

$= 1 - \phi\left(\frac{61.437 - \mu}{\sqrt{\frac{125}{25}}}\right)$

dist normal

- $\theta \in \Theta_0 \quad \mu = 60$

  $\beta(60) = 1 - \phi(0{,}64) = 0{,}26$

- $\theta \in \Theta_1 \quad \mu = 65$

  $\beta(65) = 1 - P(\text{TIPO II}) = 0{,}99$

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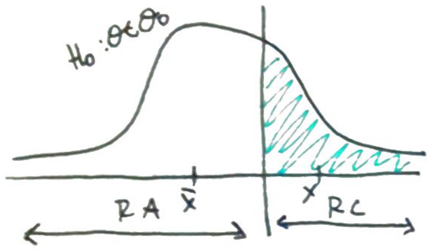
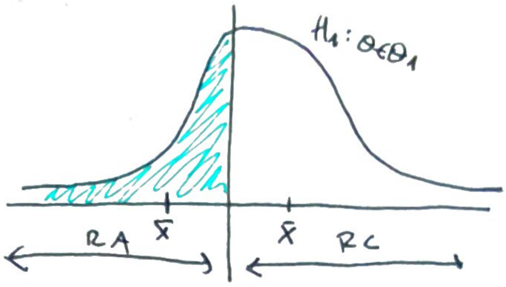
ERROR TIPO I



ERROR TIPO II



FUNCIÓN POTENCIA (BIS)

Defino una función que me calcula probabilidad

$\beta(\theta) = P(R.C) \quad ; \quad \theta \in \Theta_0 \cup \Theta_1$

- Si $\theta \in \Theta_0$, $\beta(\theta) = \underset{\theta \in \Theta_0}{P(RC)} = P(\text{Error tipo I})$

- Si $\theta \in \Theta_1$, $\beta(\theta) = \underset{\theta \in \Theta_1}{1 - P(RC)}$

* Ejercicio

Construir R.C. con un nivel de significación de $\alpha \in (0,1)$

$\begin{cases} H_0: \mu = 60 \\ H_1: \mu = 65 \end{cases}$

Una cota para los errores de tipo I

Establezco un límite (no quiero regiones críticas que me den un area mayor que (

$\boxed{\begin{array}{l} P_\theta(R.C) \leq \alpha \\ \forall \theta \in \Theta_0 \end{array}}$ Nivel de significación

R.C. $= \{\bar{x} \geq \lambda\}$ Nivel de significación de 0,1
$\forall \theta \in \Theta_0$

$\rightarrow \mu = 60 \quad P(R.C) = P(\bar{x} \geq \lambda) \leq \alpha = 0.1$

Tipificamos

↳

Tipificamos

$$\bar{x} \sim N\left(\mu = 60, \ \sigma^2/n = \frac{125}{25} = 5\right)$$

$$P(\bar{x} \geq \lambda) \leq 0{,}1$$

$$\underbrace{P\left(\frac{\bar{x}-60}{\sqrt{5}}\right.}_{N(0,1)} \geq \left.\frac{\lambda - 60}{\sqrt{5}}\right) \leq 0{,}1 \longrightarrow \underbrace{P\left(z\right.}_{N(0,1)} \geq \left.\frac{\lambda-60}{\sqrt{5}}\right) \leq 0{,}1$$

CAMBIO PARA LA

$$1 - \phi\left(\frac{\lambda-60}{\sqrt{5}}\right) \leq 0{,}1 \quad ; \quad \phi \text{ dist } N(0,1)$$

$$0{,}9 \leq \phi\left(\frac{\lambda-60}{\sqrt{5}}\right)$$ Miramos en la tabla

OTRA TABLA

menor ~~mayor~~ o igual

Y vemos que $X = 1{,}2816$; $P_{\theta \in \Theta_0}(R.C) = 0.1$

0,08	0,09	
0,897	0,9015	1.2

$\rightarrow X = 1{,}29 \Rightarrow RC = \{\bar{x} \geq 62.88\}$

$$X = \frac{\lambda - 60}{\sqrt{5}} \Rightarrow \lambda = 62{,}8845$$

$\rightarrow X = 1{,}28 \Rightarrow R.C. = \{\bar{x} \geq 62{,}86\}$

$$X = \frac{\lambda-60}{\sqrt{5}} \Rightarrow \lambda = 62{,}8621$$

$\lambda \geq 62{,}88$ Nivel de ~~sig~~ significación de 0.1

$$\boxed{\alpha = \sup_{\theta \in \Theta_0} P_\theta(R.c)}$$ Tamaño del test

Def: $\boxed{\text{P-valor}}$ lo proporciona la muestra para un determinado contraste.

$H_0: \theta \in \Theta_0$

$H_1: \theta \in \Theta_1$

$R.C. = \{\vec{x} : T(\vec{x}) \geq c\}$

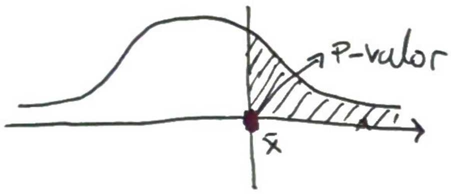
T estadístico; c. conocida

$t(\vec{x}) = \vec{X}$

$c = T(\vec{x}) = \vec{x}$

$P_{\theta} (R.C) \leq \alpha$ nivel de significación
$\theta \in \Theta_0$

$$\sup_{\theta \in \Theta_0} P_{\theta} (T(\vec{x}) \geq T(\vec{x})) = p(x_1, \dots x_n) \rightarrow \boxed{\text{p-valor}}$$



↳ p-valor cualquier otra muestra $\bar{x}'$ sea mayor igual que $\bar{x}$

↳ Probabilidad de que $\bar{x}$ sea mayor igual que el valor de mi muestra $\bar{x}$ es muy pequeña

TEST DE LA RAZÓN DE VEROSIMILITUDES

(Método para construir las R.C. del constrante)

Para contrastar $H_0: \theta \in \Theta$ frente a $H_1: \theta \in \Theta_1$ con m.a.s de tamaño n de una población $f_{\theta}(x)$

$$\boxed{R.C. = \{\vec{x} : \lambda(\vec{x}) \leq k\}}$$

$$\lambda(\vec{x}) = \frac{\sup_{\theta \in \Theta_0} f_{\theta}(\vec{x})}{\sup_{\theta \in \Theta = \Theta_0 \cup \Theta_1} f_{\theta}(\vec{x})}$$

k constante, $0 \leq k \leq 1$

α valor fijo $\alpha \in [0, 1]$

$\sup_{\theta \in \Theta_0} P(R.C) \leq \alpha$ nivel de significación

※ Ejemplo

Bernoulli (θ) $\quad \theta \in [0,1]$

$$\begin{cases} H_0: \theta \leq \theta_0 \\ H_1: \theta > \theta_0 \end{cases}$$

Test de la R.V. dado α

$$f_\theta(\vec{x}) = \theta^{\sum^n x_i} (1-\theta)^{n - \sum_{i=1}^n x_i}$$

$$\ln f_\theta(\vec{x}) = \left(\sum^n x_i\right) \ln\theta + \left(n - \sum_i^n x_i\right) \ln(1-\theta)'$$

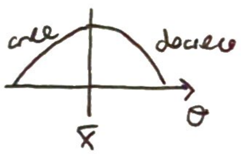
busco el máximo

$$\frac{\partial}{\partial\theta} f_\theta(\vec{x}) = \frac{\sum x_i}{\theta} - \left(n - \sum x_i\right)\frac{1}{1-\theta} = 0$$

$$\frac{\left(\sum x_i\right)}{\theta} = \left(n - \sum x_i\right)\frac{1}{1-\theta}$$

Despejando:

$\rightarrow \hat{\theta} = \bar{X}$



- Numerador

$\theta \in \Theta_0$; $\theta \leq \theta_0$

$\theta > \bar{x} \rightarrow$ Máximo $\bar{x}$

$\theta < \bar{x} \rightarrow$ Máximo $\bar{\theta}$

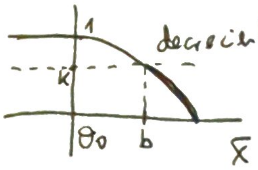
- Denominador

siempre $\bar{X}$

$$\rightarrow \lambda(x) = \begin{cases} 1 & \text{si } \theta_0 \geq \bar{X} \\ \dfrac{\theta_0^{n\bar{x}}(1-\theta_0)^{n-n\bar{x}}}{\bar{X}^{nx}(1-\bar{x})^{n-nx}} & \text{si } \theta_0 < \bar{X} \end{cases}$$

↪

Estudio $\lambda(\bar{x})$



decreciente porque $\frac{\lambda}{\partial \bar{x}} < 0$

R.C. $\{\bar{x} : \lambda(\bar{x}) \leq k\} \longrightarrow$ R.C. $= \{\bar{x} : \bar{x} \geq b\}$

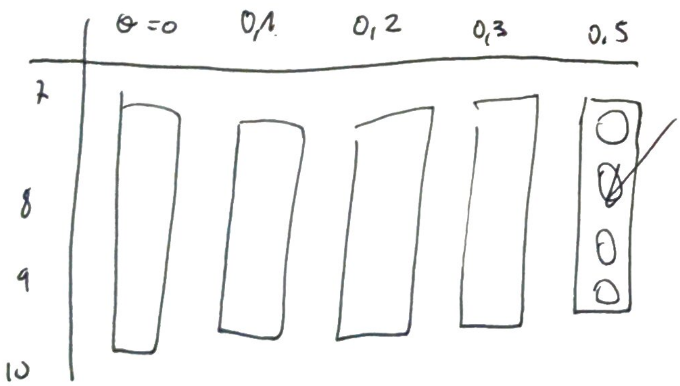
\# Pregunta ej anterior

CASO $\theta_0 = 1/2$, 7 éxitos, 10 experimentos Bernoulli

p-valor $= \underset{\theta \in \Theta_0}{P}(\bar{X} \geq \bar{x}) = \underset{\substack{\theta \in \Theta_0 \\ n=10}}{P}(\sum X_i \geq \sum x_i) = P(\underbrace{\sum X_i}_{\text{Binomial}(10,\theta)} \geq 7) =$

$= P(Z = 7) + P(Z=8) + P(Z=9) + P(Z=10)$

$H_0 : \theta \leq 1/2$ miro en la tabla



Cojo estos porque tienen el máximo y los sumo

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Ejemplo

Bernoulli (θ)
$\theta \in [0,1]$
Test R.V.

$$\begin{cases} H_0 : \theta \in \Theta_0 \\ H_1 : \theta \in \Theta_1 \end{cases}$$

$$f_\theta(\vec{x}) = \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{n-\sum_{i=1}^{n} x_i}$$

$$\ln f_\theta(\vec{x}) = \left(\sum_i^n x_i\right) \ln\theta + \left(n - \sum_i^n x_i\right) \ln(1-\theta)$$

$$\frac{\partial}{\partial\theta} \ln f_\theta(\vec{x}) = \frac{(\sum x_i)}{\theta} - \frac{(n - \sum^n x_i)}{1-\theta} = 0 \Rightarrow \hat{\theta} = \bar{x}$$

Maxim global dcha

→ Numerador $\sup_{\theta\in\Theta_0} f_\theta(\vec{x})$ $\theta = \theta_0$

$$f_{\theta_0}(\vec{x}) = \theta_0^{\sum^n x_i} (1-\theta_0)^{n-\sum^n x_i}$$

→ Denominador $\sup_{\theta\in\Theta_0\cup\Theta_1} f_\theta(\vec{x})$ (máximo para $\bar{x}$)

Si $\bar{x} = \theta_0 \Rightarrow \lambda(\bar{x}) = 1$

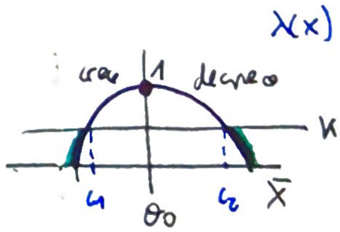
Si $\bar{x} \neq \theta_0 \Rightarrow \lambda(\bar{x}) = \left(\frac{\theta_0}{\bar{x}}\right)^{n\bar{x}} \left(\frac{1-\theta_0}{1-\bar{x}}\right)^{n-n\bar{x}}$

→ R.C. = $\{\vec{x} : \lambda(\bar{x}) \le k\}$

$$\ln\lambda(\bar{x}) = n\bar{x}(\ln\theta_0 - \ln\bar{x}) + n(1-\bar{x})\cdot(\ln(1-\theta_0) - \ln(1-\bar{x}))$$

$$\frac{\partial}{\partial\bar{x}} \ln\lambda(\bar{x}) = n\cdot\ln\left(\frac{\theta_0}{1-\theta_0}\right) - n\ln\left(\frac{\bar{x}}{1-\bar{x}}\right) \le 0 \Rightarrow n\ln\left(\frac{\theta_0}{1-\theta_0}\right) \le n\ln\left(\frac{\bar{x}}{1-\bar{x}}\right)$$

$$\frac{\theta_0}{1-\theta_0} \le \frac{\bar{x}}{1-\bar{x}} \leftrightarrow \frac{1-\bar{x}}{\bar{x}} \le \frac{1-\theta_0}{\theta_0} \Rightarrow \theta_0 \le \bar{x}$$



$$\boxed{R.C. = \{\bar{x} : \bar{x} \le c_1,\ \bar{x} \ge c_2\}}$$

→ Det c_1, c_2

→ calcular K

$K = \lambda(c_1) = \lambda(c_2)$ constantes del test

Ejercicio

Calcular k para que el test tenga tamaño $\alpha = 0{,}05$; $\theta_0 = 1/2$; $n = 1000$

Tamaño test

↳ $\sup_{\theta \in \Theta_0} P(R.C.)$

para calcular con una binomial

$$R.C. = \{ \bar{x} : \bar{x} \leq c_1,\ \bar{x} \geq c_2 \} = \{ \bar{x} : \underbrace{\sum^n x_i \leq n c_1}_{\sim N(0,1)},\ \sum^n x_i \geq n c_2 \}$$

por tanto c simétrico?

$$\rightarrow 0{,}05 = \sup_{\theta \in \Theta_0} P_\theta (R.C.) = P_{\theta = 1/2} \left(|\bar{X} - 1/2| \geq |c - 1/2| \right)$$

c_1 c_2

→ TEOREMA CENTRAL LÍMITE (muestra grande $n >>> 30$)

$$\frac{\bar{X} - E[\bar{x}]}{\sqrt{V[\bar{x}]}} \xrightarrow{\mathcal{L}} N(0,1)$$

$$E[\bar{x}] = \theta$$

$$V[\bar{x}] = \frac{1}{n} V(X) = \frac{1}{n}\theta(1-\theta) \Rightarrow \sqrt{V(\bar{x})} = 0{,}0158$$

$$P_{\theta = 1/2} \left(\underbrace{\frac{|\bar{X} - 0{,}5|}{0{,}0158}}_{\sim N(0,1)} \geq \frac{|c - 0{,}5|}{0{,}0158} \right) = 1 - P\left(\frac{|\bar{X} - 0{,}5|}{0{,}0158} \leq \overbrace{\frac{|c - 0{,}5|}{0{,}0158}}^{d} \right) = 1 - P(-d \leq N(0,1) \leq d)$$

$$= 1 - [P(N(0,1) \leq d) - \underbrace{P(N(0,1) \leq -d}_{\underbrace{P(N(0,1) \geq d}_{(1 - (N(0,1) \leq d)}}] =$$

$$= 1 - 2P(N(0,1) \leq d) + 1 = -2P(N(0,1) \leq d) + 2 = 0{,}05$$

$$P(N(0,1) \leq d) = 0{,}975 \Rightarrow \phi(d) = 0{,}975 \ ; \ \boxed{d = 1{,}96}$$

$$d = \frac{c - 0{,}5}{0{,}0158} \Rightarrow c = 0{,}53$$

$$\boxed{R.C = \{ \bar{x} \mid \bar{x} \leq -0{,}53,\ \bar{x} \geq 0{,}53 \}}$$

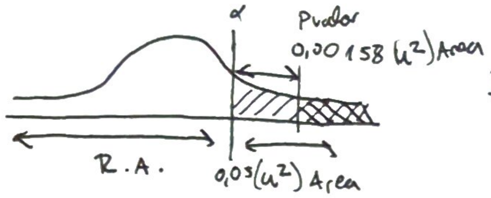
↳ Calcular el P-valor

calcular el p-valor

$$\text{p-valor} = \sup_{\theta\in\Theta_0} \dashrightarrow P_{\theta=1/2}\left(|\bar{x}-0{,}5| \geq |\bar{x}-0{,}5|\right) = P_{\theta=1/2}\left(\frac{|\bar{x}-0{,}5|}{0{,}0158} \geq \underbrace{\frac{|\bar{x}-0{,}5|}{0{,}0158}}_{3'16}\right) =$$

$$= P\left(|N(0,1)| \geq 3.16\right) = 2-2P\left(N(0,1) \leq 3.16\right) =$$

$$= 2-\phi(3.16) = 2-2\,0.99921 = 0.00158$$



Hay mucha distancia entre α y el p-valor eso es bueno para rechazar la evidencia
"Tenemos mucha evidencia de que podemos rechazar"

※ Ejemplo

$H_0: \theta \leq 1$ M.a.s. $n=1$ $x=3$ Poisson (θ)

$H_1: \theta > 1$

$\theta \in (0,2)$

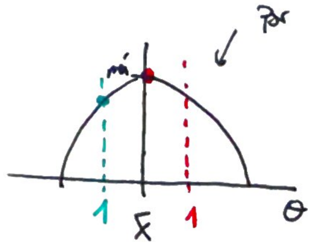
Función densidad poisson

$$f_\theta(\vec{x}) = \frac{e^{-n\theta}\,\theta^{\sum x_i}}{\prod_{i=1}^{n} x_i} \qquad x=0,1,2,\dots$$

$$\ln f_\theta(\vec{x}) = -\ln \prod x_i - n\theta + \left(\sum x_i\right)\ln\theta$$

$$\frac{\partial}{\partial\theta}\ln f_\theta(\vec{x}) = -n + \frac{\sum x_i}{\theta} = 0 \rightarrow n = \frac{\sum x_i}{\theta} \Rightarrow \hat{\theta} = \bar{x}$$

$$\frac{\partial^2}{\partial\theta^2}\ln f_\theta(\vec{x}) = -\frac{\sum x_i}{\theta^2} < 0 \qquad \hat{\theta} = \bar{x} \text{ máximo}$$



Por la primera deriv

$\frac{\partial}{\partial\theta}\ln f_\theta$ (

Numerador: $\sup_{\theta\in\Theta_0} f_\theta(\vec{x}) =$

$\theta \leq 1$

$(0,1]$

$\begin{cases} \text{Si: } \bar{x} > 1 & \text{max es } 1 \\ \text{Si: } \bar{x} < 1 & \text{máx es } \bar{x} \end{cases}$

Denominador

$\bar{x}$

Construimos $\lambda(x)$

$$\lambda(x) = \begin{cases} \text{Si } \bar{x} \leq 1 & 1 \\ \text{Si } \bar{x} > 1 & \dfrac{e^{-n\cdot 1}\, 1^{(\sum x_i)}}{e^{-n\bar{x}}\, \bar{x}^{(\sum x_i)}} \end{cases}$$

sustituyo por $\frac{1}{\bar{x}}$ en $f_\theta($

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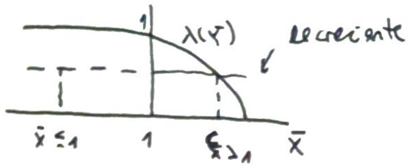
$$\lambda(\bar{x}) = \begin{cases} \text{Si } \bar{x} < 1 & 1 \\ \text{Si } \bar{x} \geq 1 & e^{-n(\bar{x}-1)} / \bar{x}^{n\bar{x}} \end{cases}$$

Estudiamos $\lambda(\bar{x})$:



$RC = \{\vec{x} / \lambda(\bar{x}) \leq k\} = \{\bar{x} / \bar{x} \geq c\} = \{x : x \geq c\}$ Poisson(θ)

$\lambda(c) = k$

General: $\sum^{n} x_i \sim$ Poisson $(n\theta)$

$x_i \sim$ Poisson (θ)

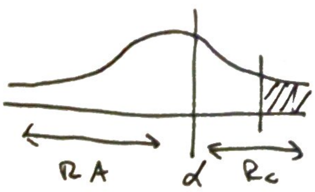
Tamaño α, $\alpha = \sup_{\theta \in \Theta_0} P(RC)$; p-valor: $\sup_{\theta \in \Theta_0} P(X \geq x) = \sup_{\theta \in (0,1]} P(X \geq 3)$

$P(X \geq 3) = 1 - P(X \leq 3) = 1 - [\, P(X=0) + P(X=1) + P(X=2) \,]$

$\sup_{\theta \in (0,1]}$: $P(X = 0 \mid \theta = 1) = 0'3679$

$P(X = 1 \mid \theta = 1) = 0'3679$

$P(X = 2 \mid \theta = 1) = 0'1839$



$\longrightarrow \sup_{\theta \in \Theta_0} P(X \geq 3) = 0'0803$

※ Ejemplo

$X_1, \dots, X_n$ población gamma $(a, p=1)$ $\quad a \in (0, \infty)$

Test RV

$$\begin{cases} H_0 : a = a_0 \\ H_1 : a \neq a_0 \end{cases}$$

$R.C = \{ \vec{x} \,/\, \lambda(\vec{x}) \leq k \}$

$$\lambda(\vec{x}) = \frac{\sup_{\theta \in \Theta_0} f_\theta(\vec{x})}{\sup_{\theta \in \Theta} f_\theta(\vec{x})}$$

$$f_\theta(\vec{x}) = \frac{a^{np}}{(\underbrace{\Gamma(p)}_{=1})^n} e^{-a \sum_1^n x_i} \prod_1^n x_i^{p-1}$$

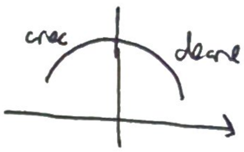
$$\ln f_\theta(\vec{x}) = n \ln a - n \ln(\Gamma(1)) - a \sum x_i + (p-1) \sum_1^n \ln x_i$$

$$\frac{\partial}{\partial \theta} \ln f_\theta(\vec{x}) = \frac{n}{a} - \left(\sum_1^n x_i\right) = 0 \Rightarrow \frac{n}{a} = \sum_1^n x_i \Rightarrow a = \frac{1}{\bar{x}}$$

$\frac{n}{a} < \sum x_i$ decreciente

$\downarrow$

$\frac{1}{\bar{x}} < a$ decreciente



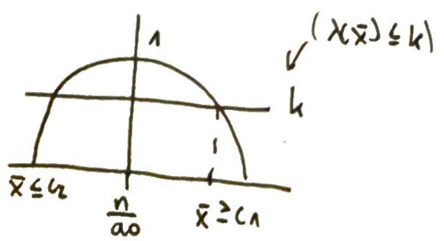
- Numerador: $a = a_0$
- Denominador: $1/\bar{x}$

$$a_0 \neq \frac{1}{\bar{x}} \to \lambda(\bar{x}) = \frac{a_0^n e^{-a_0 \bar{x}} \prod_1^n x_i^{p-1}}{\left(\frac{1}{\bar{x}}\right)^n e^{-n} \prod_1^n x_i^{p-1}} = (a_0 \bar{x})^n e^{n - a_0 \bar{x}}$$

$$\Rightarrow \lambda(\bar{x}) = \begin{cases} 1 & \text{si } 1/\bar{x} = a_0 \\ (a_0 \bar{x})^n e^{n - a_0 \bar{x}} & \text{si } \frac{1}{\bar{x}} \neq a_0 \end{cases}$$

$$\ln \lambda(\bar{x}) = n \ln(a_0 \bar{x}) + n - a_0 \bar{x}$$

$$\frac{\partial}{\partial \bar{x}} \ln(\bar{x}) = \frac{n\, a_0}{a_0 \bar{x}} - a_0 \Rightarrow \frac{n}{\bar{x}} = a_0 \Rightarrow \bar{x} = \frac{n}{a_0}$$



$RC = \{ \bar{x} : \bar{x} \geq c_1, \bar{x}_2 \leq c_2 \}$

$X_i \sim \text{gamma}(a, p=1)$

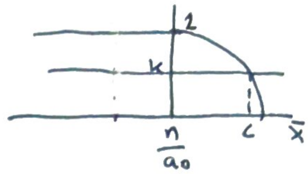
$\sum X_i \sim \text{gamma}(a, n)$

$\frac{1}{n} \sum X_i \sim \text{gamma}(na, n)$

$*^{1}$ Ejemplo 1

$$\begin{cases} H_0 : a \geq a_0 \\ H_1 : a < a_0 \end{cases}$$

$$\begin{cases} \text{Si } a_0 \leq \frac{n}{\bar{x}} , & \lambda(\bar{x}) = 1 \\ \text{Si } a_0 > \frac{n}{\bar{x}} , & \lambda(\bar{x}) = (a_0 \bar{x})^n e^{n - a_0 \bar{x}} \end{cases}$$



R.C. $= \{ \bar{x} : \bar{x} \geq c \}$

CONTRASTES SOBRE LA MEDIA POBLACIONAL NORMAL:

$*$ Ejemplo

$N(\mu, \sigma^2)$; σ^2 conocida

$$\begin{cases} H_0 : \mu = \mu_0 \\ H_1 : \mu \neq \mu_0 \end{cases} \qquad f_\theta(\vec{x}) = \frac{1}{(\sqrt{2\pi}^n \sigma^n)} e^{-\frac{1}{2\sigma^2} \sum_1^n (x_i - \mu)^2}$$

$\bar{x} \sim N(\mu, \sigma^2/n)$ $\longleftarrow$ $\ln f_\theta(\vec{x})$; $\frac{\partial}{\partial \mu} \ln (f_\theta(\vec{x})) = 0 \longrightarrow \hat{\theta} = \bar{x}$

$\frac{\bar{x} - \mu}{\sigma} \sqrt{n} \sim N(0,1)$

$$\lambda(x) = \begin{cases} 1 & \text{Si } \bar{x} = \mu_0 \\ e^{\frac{-n}{2\sigma^2} (\mu_0 - \bar{x})^2} & \text{Si } \bar{x} \neq \mu_0 \end{cases}$$

$$\ln \lambda(\bar{x}) = \frac{-n}{2\sigma^2} (\mu_0 - \bar{x})^2$$

$$\frac{\partial}{\partial x} \ln \lambda(\bar{x}) = \frac{n}{2\sigma^2} 2 (\mu_0 - \bar{x})^2 = 0 \Rightarrow \frac{n \mu_0}{\sigma^2} = \frac{n \bar{x}}{\sigma^2} \Rightarrow \mu_0 = \bar{x}$$

R.C. $= \{ \vec{x} \ / \ |\bar{x} - \mu_0| \geq |c - \mu_0| \}$

$$\alpha = \sup_{\theta \in \Theta_0} P(R.C) = \underbrace{\sup_{\theta \in \Theta_0}}_{\mu = \mu_0} P\Big(\underbrace{\sqrt{n} \frac{(\bar{x} - \mu_0)}{\sigma}}_{z} \geq \underbrace{\frac{|c - \mu_0| \sqrt{n}}{\sigma}}_{d} \Big) = P(|z| \geq d) = 1 - P(|z| \leq d)$$

$z \sim N(0,1)$

$= 1 - [P(-d \leq z \leq d)] = 2 - 2\phi(d)$; $\phi(d) = 1 - \alpha/2$

$d = z_{\alpha/2}$

* Ejemplo

$N(\mu, \sigma^2)$ σ^2 desconocido

$$\begin{cases} H_0: \mu = \mu_0 \\ H_1: \mu \neq \mu_0 \end{cases}$$

$$\hat{\theta} = \hat{x}$$

$$\lambda(\bar{x}) = \begin{cases} 1 & \text{si } \bar{x} = \mu_0 \\ e^{-\frac{n(\mu_0 - \bar{x})^2}{2\sigma^2}} & \text{si } \bar{x} \neq \mu_0 \end{cases}$$

$$\frac{\bar{x} - \mu}{\sigma} \sqrt{n} \sim N(0,1)$$

$$(n-1)\frac{s^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$\frac{\bar{x} - \mu}{s} \sqrt{n} \sim t_{n-1}$$

$$R.C. = \left\{ \bar{x} : \sqrt{n} \frac{|\bar{x} - \mu_0|}{s} \geq \frac{|c - \mu_0|}{s} \right\}$$

Igual que anterior con $d = t_{n-1, \alpha/2}$

* Ejercicio

23-05-2017

Utilizar el test de verosimilitud para hallar la RC. dado Población $N(\theta, 1)$ m.a.s. Test RV para contrastar

$$\begin{cases} H_0: \theta = 0 \\ H_1: \theta > 0 \end{cases}$$

$$f_\theta(\bar{x}) = \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}\sum_1^n (x_i - \theta)^2}$$

$$\ln f_\theta(\bar{x}) = -n \ln(\sqrt{2\pi}) - \frac{1}{2} \sum (x_i - \theta)^2$$

$$\frac{\partial \ln}{\partial \theta} f_\theta(\vec{x}) = -\frac{1}{2} \sum_1^n 2(x_i - \theta) = \sum_1^n x_i - n\theta = 0 \qquad \hat{\theta} = \sum_1^n x_i / n = \bar{x}$$

- Numerador: $\theta = 0$

$$\frac{1}{\sqrt{2\pi}} n \, e^{-\frac{1}{2} \sum x_i^2}$$

- Denominador $\frac{1}{2\pi} n \, e^{-\frac{1}{2} \sum (x_i - \bar{x})^2}$

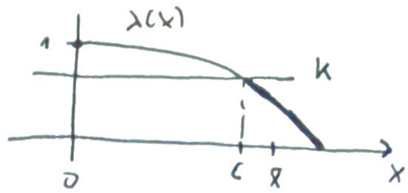
$$\lambda(\bar{x}) = \begin{cases} 1 & \text{si } \bar{x} = 0 \\ e^{-\frac{1}{2} \sum x_i^2 - \sum (x_i - \bar{x})^2} & \text{si } \bar{x} \neq 0 \end{cases}$$

$$\parallel$$

$$e^{-n\bar{x}^2/2}$$

$\nearrow$

$$\lambda(\bar{x}) = e^{-\frac{1}{2} n \bar{x}^2}$$



$$\ln \lambda(\bar{x}) = -\frac{1}{2} n \bar{x}^2$$

$$\frac{\partial}{\partial x} \ln \lambda(\bar{x}) = -\frac{2n\bar{x}}{2} \Rightarrow -n\bar{x} < 0 \,;\ \bar{x} > 0 \quad \text{decreciente}$$

$$R.C. = \{\bar{x} / \lambda(\vec{x}) \leq k\} = \{\vec{x} / \bar{x} \geq c\} \rightarrow \lambda(c) = k$$

→ Función potencia del test

Por las tablas

$$\beta(\theta) = P_{\theta\in\Theta}(R.C.) = P_{\theta\in\Theta}(\bar{x} \geq c) = 1 - P_{\theta\in\Theta}(\bar{x} \leq c) =$$

$$\bar{X} \sim N(\theta, \overset{\text{varianza}}{1/n})$$

$$= 1 - P_{\theta\in\Theta}\big(\underbrace{(\bar{x}-\theta)\sqrt{n}}_{N(0,1)} \leq (c-\theta)\sqrt{n}\big) =$$

$$= 1 - \phi((c-\theta)\sqrt{n}) \rightarrow \phi \text{ f } N(0,1)$$

$$\beta(0.1) = 0,95$$

$$\beta(-0.1) = 0,05$$

$$\theta = 0.1 \rightarrow 0.95 = 1 - \phi((c-0.1)\sqrt{n}) \Rightarrow \phi((c-0.1)\sqrt{n}) = 0,05$$

$$(c - 0.1)\sqrt{n} = -1,6449$$

$$\theta = -0.1 \rightarrow 0.05 = 1 - \phi((c+0.1)\sqrt{n}) \Rightarrow \phi((c+0.1)\sqrt{n}) = 0,95$$

$$(c+0.1)\sqrt{n} = 1,6449$$

$$\Rightarrow 2c\sqrt{n} = 0 \Rightarrow c = 0$$

$$\begin{cases} \lambda(c) = k \\ k = 1 \end{cases}$$

✱ Ejercicio

$N(\mu, \sigma^2)$ parámetro σ^2

Fijado nivel de significación α

$H_0 = \sigma^2 = 64.25$ $\qquad\qquad (n-1)\, S^2/\sigma^2 \sim \chi^2_{n-1}$

$H_1 : \sigma^2 \neq 64,25$

$n = 20, \quad \bar{x} = 81,42$

$\quad S = 87,47$

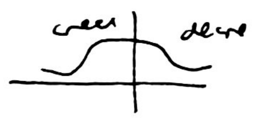
$f_\theta(\vec{x})$

$$\ln f_\theta(\vec{x}) = -n \ln \sqrt{2\pi} - n \ln \sigma - \frac{1}{2\sigma^2} \sum^n (x_i - \mu)^2$$

$$\frac{\partial}{\partial \sigma} \ln f_\theta(\vec{x}) = \frac{-n}{\sigma} + \frac{1}{\sigma^3} \sum (x_i - \mu)^2 = 0 \Rightarrow \hat{\sigma}^2 = \frac{(n-1)}{n} S^2$$

$$\frac{n}{\sigma} \leq \frac{1}{\sigma^3} \sum (x_i - \mu)^2 \text{ decreciente}$$

$$\sigma^2 \geq \frac{1}{n}(n-1) S^2 = \hat{\sigma}^2$$



• Numerador : $\theta = 64,25 \rightarrow f_\theta = 64.25(\vec{x})$

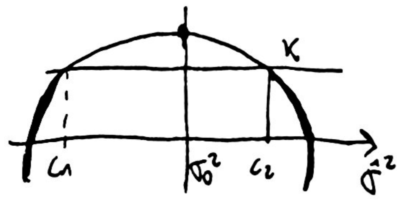
• Denominador : $\theta = \hat{\sigma}^2 = f_\theta = \hat{\sigma}^2(\vec{x})$

$$\lambda(\hat{\sigma}^2) = \begin{cases} 1 & \text{si } \hat{\sigma}^2 = 64.25 \\ \left(\frac{\hat{\sigma}}{\sigma_0}\right)^n e^{-\frac{n}{2}\left(\frac{\hat{\sigma}^2}{\sigma_0^2} - 1\right)} & \text{si } \hat{\sigma}^2 \neq \sigma_0^2 \end{cases}$$

$\ln \lambda(\hat{\sigma})$

$$\frac{\partial}{\partial \hat{\sigma}} \ln \lambda(\hat{\sigma}) = \frac{n}{\hat{\sigma}} - n \frac{\hat{\sigma}}{\sigma_0^2} = 0 \Rightarrow \hat{\sigma}^2 = \sigma_0^2$$

decreciente $\sigma_0^2 \leq \hat{\sigma}^2$



$R.C = \{ \bar{x} : \hat{\sigma}^2 \le c_1 ,\ \hat{\sigma}^2 \ge c_2 \}$

$\rightarrow \alpha$ tamaño test

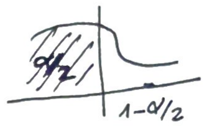
$\sup\limits_{\theta \in \Theta_0} P(R.c)$

$\theta = 64{,}25$

$P(\hat{\sigma}^2 \le c_1 ,\ \hat{\sigma}^2 \ge c_2)$

$P(\hat{\sigma}^2 \le c_1) = \alpha/2$ (como no dice nada yo tomo colas iguales)

$P(\hat{\sigma}^2 \ge c_2) = \alpha/2$



$P\Big(\underbrace{\frac{(n-1)s^2}{n\sigma_0^2}}_{\chi^2_{n-1}} \le n\, c_1/\sigma_0^2\Big) = \alpha/2 \qquad \frac{n c_1}{\sigma_0^2} = \chi^2_{n-1,\ 1-\alpha/2}$

la paso aquí

(Mirar tablas si hubiera valores)