The Finite Element Method

Section 3

Dr. Matthew Santer

Department of Aeronautics, I.C.L.

m.santer@imperial.ac.uk

Section Objectives

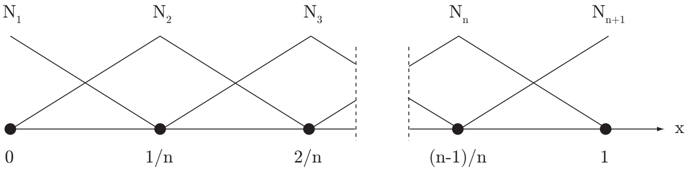
In the last section we showed how the solution to the 1-D Poisson Equation could be approximated using two degrees of freedom.

In this section we will:

- show how to move from a global to local description of the approximating functions
- introduce local descriptions of ‘finite elements’
- introduce an implementation of a local-to-global assembly operator

Finite Elements: global description

We have seen a 2-dof approximation of the 1-D domain. Accuracy of approximation may clearly be improved by increasing the number of degrees of freedom.

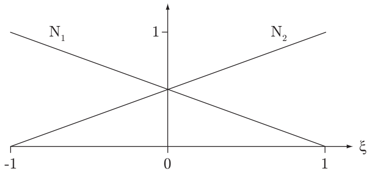


Finite Element: global description

Domain	$[x_A, x_{A+1}]$
Nodes	$\{x_A, x_{A+1}\}$
DoF	$\{d_A, d_{A+1}\}$
Shape Functions	$\{N_A, N_{A+1}\}$
Interpolation Function	$u^h(x) = N_A(x)d_A + N_{A+1}(x)d_{A+1}$

Finite Elements: local description

We notice that the individual basis functions are highly localized in space and have the same repeated form. We can therefore standardize a single local element description.



Finite Element: local description

Domain	$[\xi_1, \xi_2]$
Nodes	$\{\xi_1, \xi_2\}$
DoF	$\{d_1, d_2\}$
Shape Functions	$\{N_1, N_2\}$
Interpolation Function	$u^h(\xi) = N_1(\xi)d_1 + N_2(\xi)d_2$

Local to Global Mapping

The local coordinate ξ is related to the global coordinate x via an affine transformation (i.e. a linear transformation followed by a translation)

$$\xi : [x_A, x_{A+1}] \rightarrow [\xi_1, \xi_2]$$

such that $\xi(x_A) = \xi_1$ and $\xi(x_{A+1}) = \xi_2$.

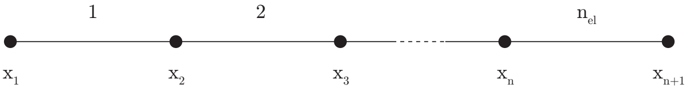
Visualizer

Determine $\xi(x)$ and $x(\xi)$ for the 2-dof finite element and consider the form of the shape functions

N.B. upper-case subscripts refer to the global element description, and lower-case subscripts refer to the local element description

Element 'Stiffness' Matrix and 'Force' Vector

Consider a finite element approximation to the 1D BVP with n_{el} elements and element numbers $e = 1, 2, \cdots, n_{el}$.



Recall that we have previously defined the global 'stiffness' matrix $\mathbf{K} = [K_{AB}]$ and 'force' vector $\mathbf{F} = \{F_A\}$ with

$$K_{AB} = a(N_A, N_B) = \int_0^1 N_{A,x} N_{B,x} dx$$

and

$$F_A = (N_A, l) + \delta_{A1} h - a(A_A, N_{n+1}) g = \int_0^1 N_A l dx + \delta_{A1} h - \int_0^1 N_{A,x} N_{n+1,x} dx g$$

Assembly: global sense

$$\mathbf{K} = \sum_{e=1}^{n_{el}} \mathbf{K}^e, \quad \mathbf{K}^e = [K^e_{AB}]$$

$$\mathbf{F} = \sum_{e=1}^{n_{el}} \mathbf{F}^e, \quad \mathbf{F}^e = \{F^e_{A}\}$$

in which

$$K^e_{AB} = a(N_A, N_B)^e = \int_{\Omega^e} N_{A,x} N_{B,x} dx$$

and

$$\begin{aligned} F^e_A &= (N_A, l)^e + \delta_{e1}\delta_{A1} h - a(A_A, N_{n+1})^e g \\ &= \int_{\Omega^e} N_A l dx + \delta_{e1}\delta_{A1} h - \int_{\Omega^e} N_{A,x} N_{n+1,x} dx g \end{aligned}$$

The elemental domain is defined as $\Omega^e = [x^e_1, x^e_2]$.

Assembly: local sense

It is much more convenient to define an individual stiffness matrix and force vector for the eth element with respect to local node numbering:

$$\mathbf{k}^e = [k_{ab}] \qquad \mathbf{f}^e = [f_a]$$

$$k^e_{ab} = a(N_a, N_b)^e = \int_{\Omega^e} N_{a,x} N_{b,x} dx$$

$$f^e_a = \int_{\Omega^e} N_a l dx + \begin{cases} \delta_{a1} h & e = 1 \\ 0 & e = 2, 3, \cdots, n_{el} - 1 \\ -k^e_{a2} g & e = n_{el} \end{cases}$$

These expressions may be determined from the global definitions by inspection.

1-D Assembly Operator Implementation

We can obtain the global stiffness matrix and force vector from the locally-defined elemental stiffness matrix and force vector by defining an assembly operator $\mathbf{A}(\cdot)$ such that

$$\mathbf{K} = \mathop{\mathbf{A}}_{e=1}^{n_{el}} (\mathbf{k}^e) \qquad \mathbf{F} = \mathop{\mathbf{A}}_{e=1}^{n_{el}} (\mathbf{f}^e)$$

This operator may be implemented computationally for the 1-D case by means of a Location Matrix (LM) which has dimensions $n_{en} \times e$.

$$\text{Global eqn. no. } A = LM(a, e) = \begin{cases} e & \text{if } a = 1 \\ e+1 & \text{if } a = 2 \end{cases}$$

Visualizer

Example assembly operation for a 1-D problem approximated with 4 elements each having 2-dof

Summary

This section has introduced one of the most important aspects of the finite element method: the local element representation.

- Local element definitions are transformed to global coordinates via an affine mapping
- An assembly operator has been introduced which generates global ''stiffness' matrices and 'force' vectors from local element stiffnesses and forces
- A computational implementation of the assembly operator for a 1-D problem has been demonstrated

1-D problems are useful to consider for fundamental understanding, but it is now time to move to the use of the finite element for 2-D domains with applicability to 'real' engineering problems.