EJERCICIOS RESUELTOS
GENTZEN PARA LÓGICA DE PREDICADOS

1) Asumiendo la validez de $\varphi \vdash \neg\neg\varphi$, demostrar:
$\{\forall x R(x) \vee \forall y \neg P(y), \neg\exists z (P(z) \wedge R(z))\} \vdash \forall y \neg P(y)$

1) $\forall x R(x) \vee \forall y \neg P(y)$ (premisa)
2) $\neg\exists z (P(z) \wedge R(z))$ (premisa)
3) $\neg(P(z_1) \wedge R(z_1))$ (T37.2 (2))
4) $\neg P(z_1) \vee \neg R(z_1)$ (T8.3 (3))

5) $\forall x R(x)$ (pr. aux)
6) $R(z_1)$ (E$\forall$ (5))
7) $\neg\neg R(z_1)$ (regla asumida (6))
8) $\neg P(z_1)$ (T29 (7,4))

9) $\forall x R(x) \rightarrow \neg P(z_1)$ (I$\rightarrow$(5,8))

10) $\forall y \neg P(y)$
11) $\neg P(z_1)$ (E$\forall$(10))

12) $\forall y \neg P(y) \rightarrow \neg P(z_1)$ (I$\rightarrow$(10,11))
13) $\neg P(z_1)$ (E$\vee$(1,9,12))
14) $\forall y \neg P(y)$ (I$\forall$(13))

2) Demostrar la regla derivada T35.1, es decir:

$\exists x(\varphi(x) \vee \psi(x)) \vdash \exists x\varphi(x) \vee \exists x\psi(x)$

1) $\exists x(\varphi(x) \vee \psi(x))$ (premisa)

2) $\varphi(x_1) \vee \psi(x_1)$ (pr. aux)

3) $\varphi(x_1)$ (p.aux)

4) $\exists x\varphi(x)$ ($I\exists(3)$)

5) $\exists x\varphi(x) \vee \exists x\psi(x)$ ($I\vee(4)$)

6) $\varphi(x_1) \to \exists x\varphi(x) \vee \exists x\psi(x)$ ($I\to(3,5)$)

7) $\psi(x_1)$ (pr. aux)

8) $\exists x\psi(x)$ ($I\exists(7)$)

9) $\exists x\varphi(x) \vee \exists x\psi(x)$ ($I\vee(8)$)

10) $\psi(x_1) \to \exists x\varphi(x) \vee \exists x\psi(x)$ ($I\to(7,9)$)

11) $\exists x\varphi(x) \vee \exists x\psi(x)$ ($E\vee(2,6,10)$)

12) $\varphi(x_1) \vee \psi(x_1) \to \exists x\varphi(x) \vee \exists x\psi(x)$ ($I\to(2,11)$)

13) $\exists x\varphi(x) \vee \exists x\psi(x)$ ($E\exists(1,12)$)

3) Demostrar la regla derivada T35.2, que es:

$\exists x \varphi(x) \vee \exists x \psi(x) \vdash \exists x (\varphi(x) \vee \psi(x))$

1) $\exists x \varphi(x) \vee \exists x \psi(x)$ (premisa)

2) $\exists x \varphi(x)$ (p.aux)

3) $\varphi(x_1)$ (p.aux)

4) $\varphi(x_1) \vee \psi(x_1)$ (I∨(3))

5) $\exists x (\varphi(x) \vee \psi(x))$ (I∃(4))

6) $\varphi(x_1) \rightarrow \exists x (\varphi(x) \vee \psi(x))$ (I→(3,5))

7) $\exists x (\varphi(x) \vee \psi(x))$ (E∃(2,6))

8) $\exists x \varphi(x) \rightarrow \exists x (\varphi(x) \vee \psi(x))$ (I→(2,7))

9) $\exists x \psi(x)$ (p.aux)

10) $\psi(x_1)$ (p.aux)

11) $\varphi(x_1) \vee \psi(x_1)$ (I∨(10))

12) $\exists x (\varphi(x) \vee \psi(x))$ (I∃(11))

13) $\psi(x_1) \rightarrow \exists x (\varphi(x) \vee \psi(x))$ (I→(10,12))

14) $\exists x (\varphi(x) \vee \psi(x))$ (E∃(9,13))

15) $\exists x \psi(x) \rightarrow \exists x (\varphi(x) \vee \psi(x))$ (I→(9,14))

16) $\exists x (\varphi(x) \vee \psi(x))$ (E∨(1,8,15))

4) Demostrar la regla derivada T36.1, que es:

$$\exists x \varphi(x) \to \exists x \psi(x) \vdash \exists x (\varphi(x) \to \psi(x))$$

1) $\exists x \varphi(x) \to \exists x \psi(x)$ (premisa)
2) $\neg \exists x \varphi(x) \lor \exists x \psi(x)$ (T9.1(1))
3) $\neg \exists x \varphi(x)$ (p. aux)
4) $\neg \varphi(y)$ (T37.2(3))
5) $\neg \varphi(y) \lor \psi(y)$ (I$\lor$(4))
6) $\varphi(y) \to \psi(y)$ (T9.2(5))
7) $\exists x (\varphi(x) \to \psi(x))$ (I$\exists$(6))
8) $\neg \exists x \varphi(x) \to \exists x (\varphi(x) \to \psi(x))$ (I$\to$(3,7))
9) $\exists x \psi(x)$ (p. aux)
10) $\psi(y)$ (p. aux)
11) $\neg \varphi(y) \lor \psi(y)$ (I$\lor$(10))
12) $\varphi(y) \to \psi(y)$ (T9.2(11))
13) $\exists x (\varphi(x) \to \psi(x))$ (I$\exists$(12))
14) $\psi(y) \to \exists x (\varphi(x) \to \psi(x))$ (I$\to$(10,13))
15) $\exists x (\varphi(x) \to \psi(x))$ (E$\exists$(9,14))
16) $\exists x \psi(x) \to \exists x (\varphi(x) \to \psi(x))$ (I$\to$(9,15))
17) $\exists x (\varphi(x) \to \psi(x))$ (E$\lor$(2,8,16))

5) Demostrar:

$\{\forall x(P(x) \to Q(x)), \forall x(P(x) \land R(x))\} \vdash \exists x(Q(x) \land R(x))$

1) $\forall x(P(x) \to Q(x))$ (premisa)
2) $\forall x(P(x) \land R(x))$ (premisa)
3) $P(a) \land R(a)$ $(E\forall(2))$
4) $P(a)$ $(E\land(3))$
5) $R(a)$ $(E\land(3))$
6) $P(a) \to Q(a)$ $(E\forall(1))$
7) $Q(a)$ $(E\to(4,6))$
8) $Q(a) \land R(a)$ $(I\land(7,5))$
9) $\exists x(Q(x) \land R(x))$ $(I\exists(8))$

6) Demostrar:

$$\{\forall x(P(x)\wedge Q(x)\to R(x)), \forall x(P(x)\wedge Q(x)\wedge R(x)\to S(x)), \exists x(S(x)\wedge P(x)\to T(x))\} \vdash \exists x(P(x)\wedge Q(x)\to T(x))$$

1) $\forall x(P(x)\wedge Q(x)\to R(x))$ (premisa)

2) $\forall x(P(x)\wedge Q(x)\wedge R(x)\to S(x))$ (premisa)

3) $\exists x(S(x)\wedge P(x)\to T(x))$ (premisa)

4) $S(y)\wedge P(y)\to T(y)$ (p.aux)

5) $P(y)\wedge Q(y)$ (p.aux)

6) $P(y)\wedge Q(y)\to R(y)$ ($E\forall(1)$)

7) $R(y)$ ($E\to(5,6)$)

8) $P(y)\wedge Q(y)\wedge R(y)\to S(y)$ ($E\forall(2)$)

9) $P(y)\wedge Q(y)\wedge R(y)$ ($I\wedge(5,7)$)

10) $S(y)$ ($E\to(8,9)$)

11) $P(y)$ ($E\wedge(9)$)

12) $S(y)\wedge P(y)$ ($I\wedge(10,11)$)

13) $T(y)$ ($E\to(4,12)$)

14) $P(y)\wedge Q(y)\to T(y)$ ($I\to(5,13)$)

15) $\exists x(P(x)\wedge Q(x)\to T(x))$ ($I\exists(14)$)

16) $(S(y)\wedge P(y)\to T(y))\to \exists x(P(x)\wedge Q(x)\to T(x))$ ($I\to(4,15)$)

17) $\exists x(P(x)\wedge Q(x)\to T(x))$ ($E\exists(3,16)$)

7) Demostrar $\exists x(P(y) \wedge Q(x)) \vdash P(y) \wedge \exists x Q(x)$

1) $\exists x (P(y) \wedge Q(x))$ (premisa)

2) $P(y) \wedge Q(z)$ (p. aux)
3) $P(y)$ $(E\wedge(2))$
4) $Q(z)$ $(E\wedge(2))$
5) $\exists x Q(x)$ $(I\exists(4))$
6) $P(y) \wedge \exists x Q(x)$ $(I\wedge(3,5))$

7) $P(y) \wedge Q(z) \rightarrow P(y) \wedge \exists x Q(x)$ $(I\rightarrow(2,6))$
8) $P(y) \wedge \exists x Q(x)$ $(E\exists(1,7))$

8) Demostrar $\forall x P(x) \vee \forall x Q(x) \vdash \forall x (P(x) \vee Q(x))$

1) $\forall x P(x) \vee \forall x Q(x)$ (premisa)

2) $\forall x P(x)$ (p.aux)

3) $P(z)$ $(E\forall(2))$

4) $P(z) \vee Q(z)$ $(I\vee(3))$

5) $\forall x P(x) \rightarrow P(z) \vee Q(z)$ $(I\rightarrow(2,4))$

6) $\forall x Q(x)$ (p.aux)

7) $Q(z)$ $(E\forall(6))$

8) $P(z) \vee Q(z)$ $(I\vee(7))$

9) $\forall x Q(x) \rightarrow P(z) \vee Q(z)$ $(I\rightarrow(6,8))$

10) $P(z) \vee Q(z)$ $(E\vee(1,5,9))$

11) $\forall x (P(x) \vee Q(x))$ $(I\forall(10))$

9) Demostrar:

$$\{\exists x(P(x)\wedge Q(x,a)), \forall x(Q(x,a)\to R(x))\}\vdash \exists x(P(x)\wedge R(x))$$

1) $\exists x(P(x)\wedge Q(x,a))$ (premisa)

2) $\forall x(Q(x,a)\to R(x))$ (premisa)

3) $P(y)\wedge Q(y,a)$ (p.aux)

4) $Q(y,a)$ ($E\wedge(3)$)

5) $Q(y,a)\to R(y)$ ($E\forall(2)$)

6) $R(y)$ ($E\to(4,5)$)

7) $P(y)$ ($E\wedge(3)$)

8) $P(y)\wedge R(y)$ ($I\wedge(6,7)$)

9) $\exists x(P(x)\wedge R(x))$ ($I\exists(8)$)

10) $P(y)\wedge Q(y,a)\to\exists x(P(x)\wedge R(x))$ ($I\to(3,9)$)

11) $\exists x(P(x)\wedge R(x))$ ($E\exists(1,10)$)

10) Demostrar:

$\{\exists x(P(x) \vee Q(x)), \forall x(P(x) \to R(x,x))\} \vdash \neg\exists x Q(x) \to \exists x R(x,x)$

1) $\exists x(P(x) \vee Q(x))$ (premisa)
2) $\forall x(P(x) \to R(x,x))$ (premisa)
3) $\neg\exists x Q(x)$ (premisa auxiliar)
4) $\neg Q(y)$ (T37.2(3))
5) $P(y) \vee Q(y)$ (pr. aux)
6) $P(y)$ (T29(4,5))
7) $P(y) \to R(y,y)$ (E∀(2))
8) $R(y,y)$ (E→(6,7))
9) $\exists x R(x,x)$ (I∃(8))
10) $P(y) \vee Q(y) \to \exists x R(x,x)$ (I→(5,9))
11) $\exists x R(x,x)$ (E∃(1,10))
12) $\neg\exists x Q(x) \to \exists x R(x,x)$ (I→(3,11))

11) Demostrar:

$$\{\forall x(P(x) \vee Q(x)), \exists x \neg Q(x), \forall x(R(x) \to \neg P(x))\} \vdash \exists x \neg R(x)$$

1) $\forall x(P(x) \vee Q(x))$ (premisa)
2) $\exists x \neg Q(x)$ (premisa)
3) $\forall x(R(x) \to \neg P(x))$ (premisa)
4) $\neg Q(y)$ (p. aux)
5) $P(y) \vee Q(y)$ ($E\forall(1)$)
6) $P(y)$ ($T29(4,5)$)
7) $R(y) \to \neg P(y)$ ($E\forall(3)$)
8) $\neg P(y)$ (p. aux)
9) $P(y) \wedge \neg P(y)$ ($I\wedge(6,8)$)
10) $\neg P(y) \to P(y) \wedge \neg P(y)$ ($I\to(8,9)$)
11) $\neg\neg P(y)$ ($I\neg(10)$)
12) $\neg R(y)$ ($T28(7,11)$)
13) $\exists x \neg R(x)$
14) $\neg Q(y) \to \exists x \neg R(x)$ ($I\to(4,13)$)
15) $\exists x \neg R(x)$ ($E\exists(2,14)$)