The Finite Element Method

Section 5

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Section Objectives

Having demonstrated a similar approach to the one followed for the simple 1-D problem for approximating a 2-D BVP with a scalar field, we will go on to consider a possible implementation.

In this section we will:

- show (as for the 1-D problem) the switch from global to local finite element persepctive
- illustrate (again building on the 1-D method) a computational implementation of the assembly operator
- illustrate this assembly operation by example

2-D BVP Galerkin Form

By following the $(S) \to (W) \to (G) \to (M)$ process we have shown that the 2-D BVP with scalar field may be approximated according to:

for $A \in \eta - \eta_g$

$$a\left(N_A, \sum_{B \in \eta - \eta_g} N_B\right) d_B = (N_A, l) + (N_A, h)_\Gamma - \sum_{B \in \eta_g} a(N_A, N_B) g_B$$

or more concisely:

$$\mathbf{Kd} = \mathbf{F}$$

or (more helpfully from the point of view of implementation):

$$[K_{PQ}]\{d_Q\} = \{F_P\} \qquad 1 \leq P, Q \leq n_{eq}$$

Global Equation Numbering

One way to ensure that the global ordering of the equations is correct is to use an ID array

$$ID(A) = \begin{cases} P & \text{if} \quad A \in \eta - \eta_g \\ 0 & \text{if} \quad A \in \eta_g \end{cases}$$

in which P is the global equation number.

This means that for nodes where g is prescribed, the equation number is assigned to zero. Hence

$$[K_{PQ}] = a(N_A, N_B) \qquad P = ID(A), \; Q = ID(B)$$

and

$$F_P = (N_A, l) + (N_A, h)_\Gamma - \sum_{B \in \eta_g} a(N_A, N_B) g_B$$

N.B. **K** is symmetric and positive-definite.

Elemental Definitions: Global Perspective

The 'stiffness' matrix **K** and 'force' vector **F** may be obtained by summing the contributions from all the individual elements.

We first consider the assembly from a global perspective. In this case we can write

$$\mathbf{K} = \sum_{e=1}^{n_{el}} \mathbf{K}^e \qquad \text{where} \qquad \mathbf{K}^e = [K^e_{PQ}]$$

and

$$\mathbf{F} = \sum_{e=1}^{n_{el}} \mathbf{F}^e \qquad \text{where} \qquad \mathbf{F}^e = [F^e_P]$$

The individual terms of the stiffness matrix and force vector are therefore

$$K^e_{PQ} = a(N_A, N_B)^e = \int_{\Omega^e} (\nabla N_A)^T \kappa (\nabla N_B) d\Omega$$

and

$$F^e_P = \int_{\Omega^e} N_A l d\Omega + \int_{\Gamma^e_h} N_A h d\Gamma - \sum_{B \in \eta_g} a(N_A, N_B)^e g_B$$

Elemental Definitions: Local Perspective

From the global descriptions we can deduce the local descriptions:

$$\mathbf{k}^e = [k^e_{ab}] \quad \mathbf{f}^e = [f^e_a] \quad 1 \leq a, b \leq n_{el}$$

in which

$$k^e_{ab} = a(N_a, N_b)^e = \int_{\Omega^e} (\nabla N_a)^T \kappa (\nabla N_b) d\Omega$$

and

$$f^e_a = \int_{\Omega^e} N_a l d\Omega + \int_{\Gamma^e_h} N_a h d\Gamma - \sum_{b=1}^{n_{en}} k^e_{ab} g^e_b$$

In this case, we define $g^e_b = g(\mathbf{x}^e_b)$ if it is prescribed at node b, and zero otherwise.

As before the local element contributions are assembled into the global stiffness matrix and force vector via an assembly operator

$$\mathbf{K} = \mathop{\mathbf{A}}_{e=1}^{n_{el}} (k^e_{ab}) \qquad \mathbf{F} = \mathop{\mathbf{A}}_{e=1}^{n_{el}} (f^e_a)$$

Standard Form

As a brief aside it is worth rewriting these local definitions in a more standard form that you may find easier to interpret.

Visualizer

Standard notation for the elemental **k** matrix:

$$\mathbf{k}^e = \int_{\Omega^e} \mathbf{B}^T \mathbf{D} \mathbf{B} d\Omega$$

We now go on to consider a detailed implementation of the assembly operator to achieve the global 'stiffness' matrix

$$\mathbf{K} = \underset{e=1}{\overset{n_{el}}{\mathbf{A}}} \left(k^e_{ab} \right)$$

Assembly Implementation

When implementing the assembly operator $\overset{n_{el}}{\underset{e=1}{\mathbf{A}}} (\cdot)$ we need to ensure that local nodes are correctly associated with global node numbers and that global equations are correctly ordered.

First, the element nodal data array – which relates local to global node numbers – is defined as

$$IEN(a, e) = A$$

in which a is the local node number, e is the element number, and A is the global node number. As previously seen, the Destination array is defined as

$$ID(A) = P \quad \text{if} \quad A \in \eta - \eta_g \quad \text{else} \quad 0$$

in which P is the global equation number. Finally, the location matrix is defined as

$$LM(a, e) = ID(IEN(a, e))$$

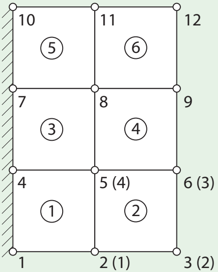
Assembly Examples

The creation and use of the location matrix to carry out the element-wise assembly is best illustrated by example.

Visualizer

Determine the location matrix *LM* ...

... and hence the terms in the global **K** matrix and **F** vector.



Summary

In this section we have shown how the 2-D BVP finite element-based approximation may be implemented. In particular we have:

- determined local finite element forms for 2-D heat conduction
- introduced a possible implementation method of the assembly operator for this problem (there are other equally-valid options)
- demonstrated the element assembly process by example

The next step is to consider the implementation of the individual finite elements in greater detail.