Topic 3: Signal Representation in Communication Systems

Telecommunication Systems Fundamentals

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Concepts in this Chapter

- ***Signal Classifications***
- ***Band-Pass Signals and Systems***
- ***Representations:***
  - ***Complex Envelope***
  - ***Base-Band Equivalent***
  - ***Discrete Base-Band Equivalent***

Theory classes: 4 sessions (8 hours)
Problems resolution: 1 session (2 hours)

Bibliography

Communication Systems Engineering. John. G. Proakis. Prentice Hall

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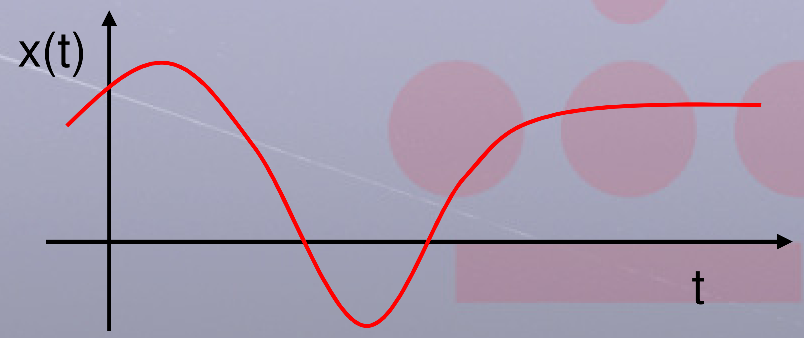
Signal Classifications

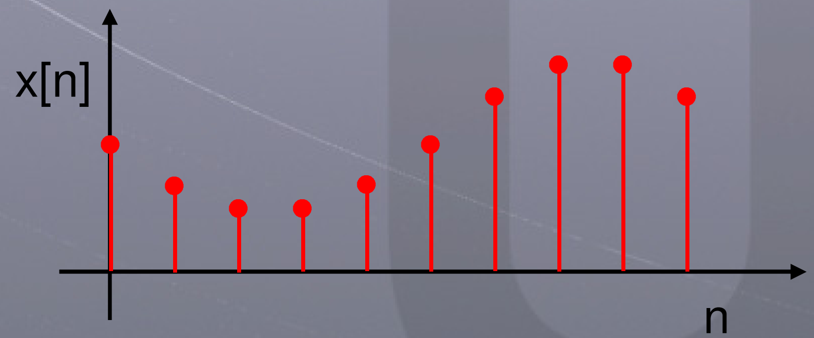
- Continious Time / Discrete Time
  - Analogue signals / Digital Signals (amplitude)
- Stochastic (random) / Deterministic signals
  - For Deterrministic: finite energy / power defined.
    - Por Power Defined: periodic / non-Periodic
- Time Limited / Time Unlimited
  - For Time Unlimited: Periodic / Non-Periodic
- Other Classifications:
  - Real valued / Complex valued
  - Even / Odd
  - Hermitian / Anti-Hermitian

Continuous & Discrete-Time Signals

- **Continuous-Time Signals**
  - Most signals in the real world are continuous time, as the scale is infinitesimally fine.
  - Eg voltage, velocity,
  - Denote by $x(t)$, where the time interval may be bounded (finite) or infinite



- **Discrete-Time Signals**
  - Some real world and many digital signals are discrete time, as they are sampled
  - E.g. pixels, daily stock price (anything that a digital computer processes)
  - Denote by $x[n]$, where n is an integer value that varies discretely



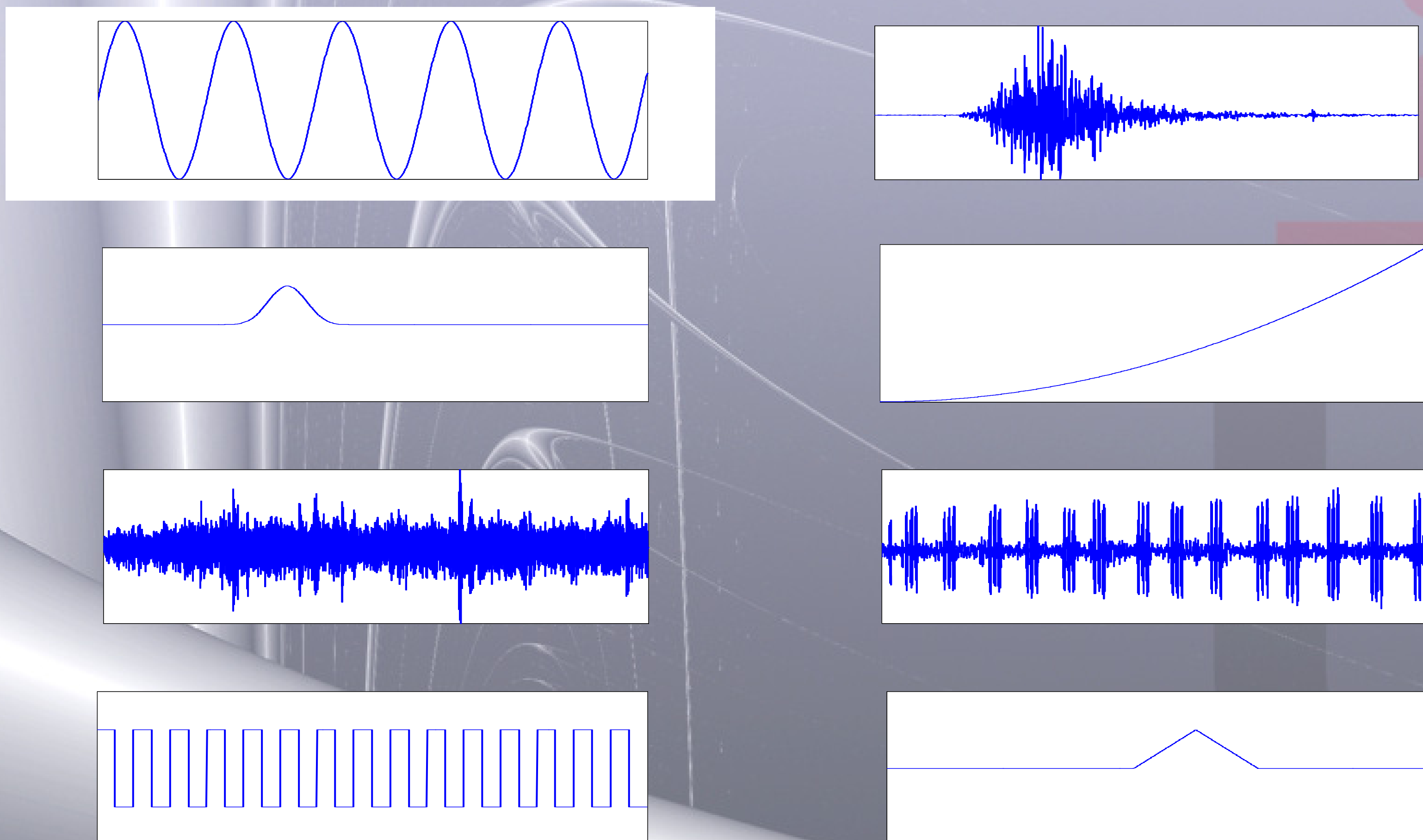
- **Sampled continuous signal**
  $x[n] = x(nk)$ – k is sample time

Signal Properties

- **Periodic signals**:
  - a signal is periodic if it repeats itself after a fixed period T, i.e. $x(t) = x(t+T)$ for all t. A $\sin(t)$ signal is periodic.
- **Even and odd signals**:
  - a signal is even if $x(-t) = x(t)$ (i.e. it can be reflected in the axis at zero). A signal is odd if $x(-t) = -x(t)$. Examples are $\cos(t)$ and $\sin(t)$ signals, respectively.
- **Exponential and sinusoidal signals**:
  - a signal is (real) exponential if it can be represented as $x(t) = Ce^{at}$. A signal is (complex) exponential if it can be represented in the same form but C and a are complex numbers.
- **Step and pulse signals**:
  - A pulse signal is one which is nearly completely zero, apart from a short spike, $\delta(t)$. A step signal is zero up to a certain time, and then a constant value after that time, $u(t)$.

- These properties define a large class of tractable, useful signals and will be further considered in the coming lectures

Stochastic & Deterministic Signals



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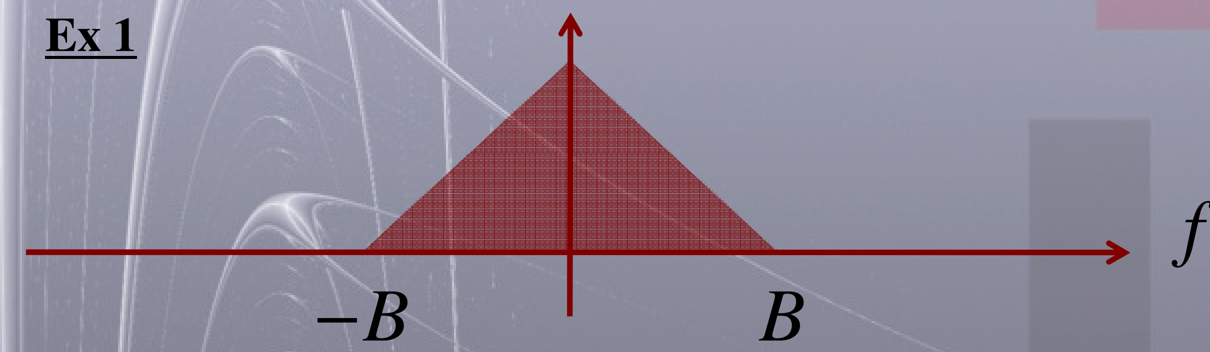
Band-Limited Signals

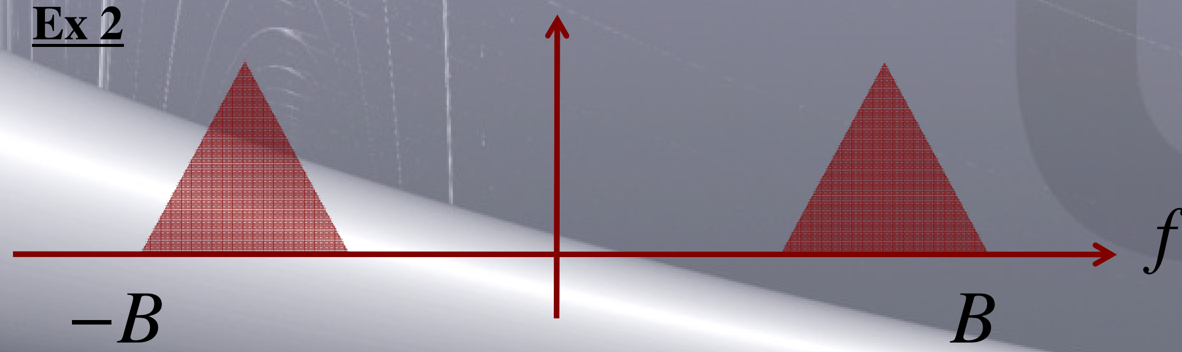
- We claim a signal is band-limited if its power/energy spectral density is non-zero on a limited frequency-range

$$S_x(f) = 0, \quad \text{if} \quad |f| \leq B$$

Ex 1



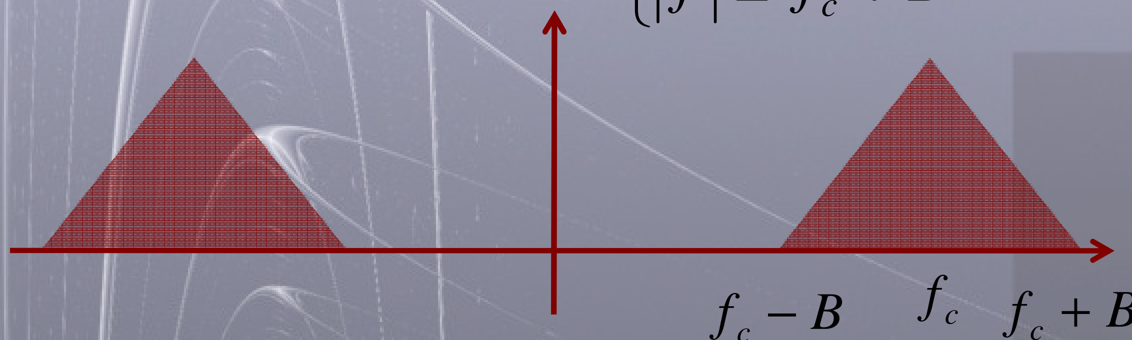
Ex 2



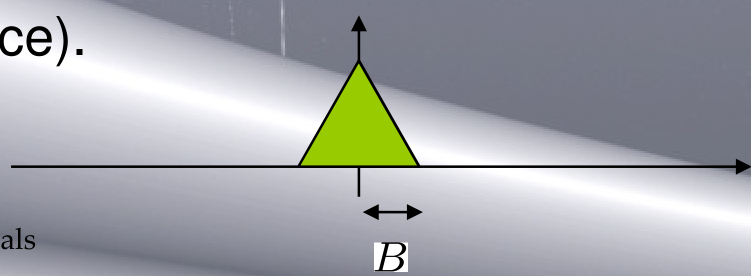
Band-Limited Signals (II)

- A band limited signal is considered band-pass if its power/energy spectral density is confined into a limited frequency range (bandwidth) around a give central frequency

$$S_x(f) = 0, \quad \text{if} \begin{cases} |f| \le f_c - B \\ |f| \le f_c + B \end{cases}$$

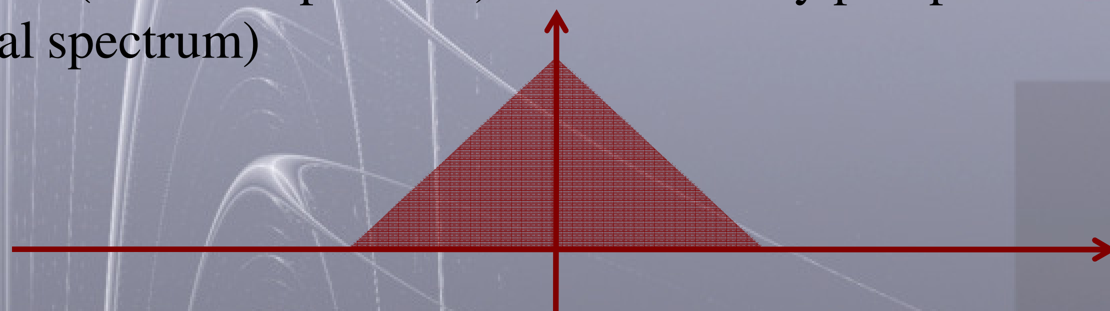


- However there are many other band limited signals which their non-zero sepectrum is around DC – we call them low-pass signals (ex. human voice).



Bilateral and Unilateral Spectrum

- Real-valued signals are known to have hermitian spectrum, i.e. the positive values of the spectrum are equal to the complex conjugates of their negative counterparts – the spectrum is symmetric. So, if the signal is known to be real-valued, we can plot both positive and negative frequencies (bilateral spectrum) or we can only plot positive frequencies (unilateral spectrum)

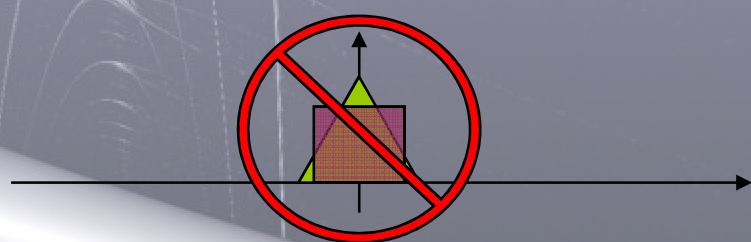


Bilateral and Unilateral Spectrum

- Both bilateral and unilateral spectrum for real-value signals are equivalente, and provide exactly the same information.
- If we use unilateral spectrum and we want to mantain the same overal power of the signal, we have to multiply by two the amplitude of the unilateral spectrum respect to the biliateral one.
- Unilateral spectrum for complex-value signals do not make sense

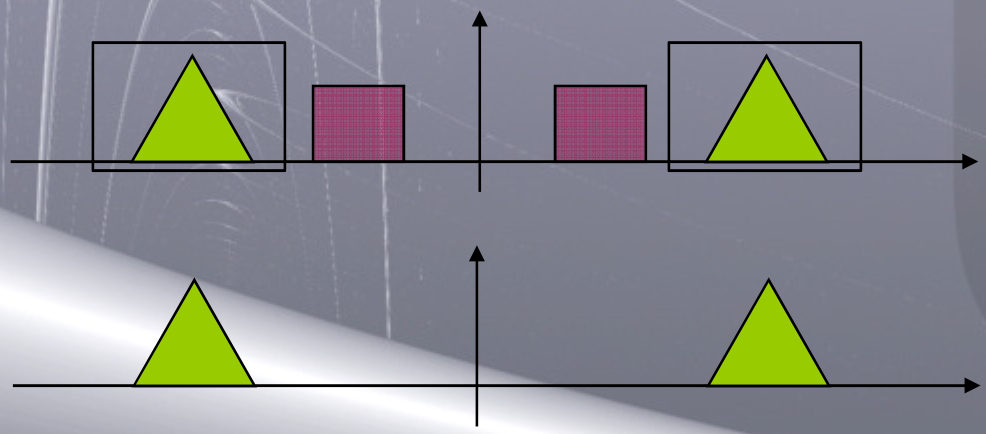
Transmission at Low Frequencies

- Some of the problems encountered when transmitting at low frequencies are:
  - Electromagnetic devices may have sizes comparable to the wavelength, so it size will increase for low frequencies
  - The noise and distortion for low frequencies usually is larger
  - It is not possible to simultaneously transmit several signals by differentiating their central frequency (FDMA)
  - Many electromagnetic channels stop low frequencies



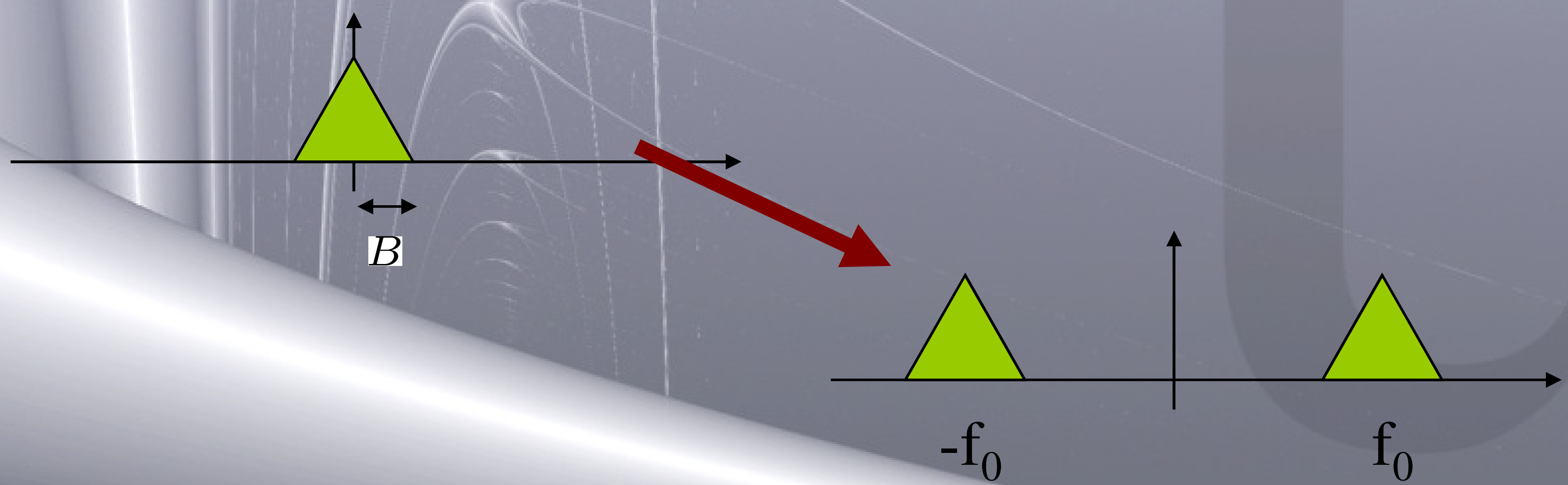
Transmission at High Frequencies

- On the other hand, transmission at high frequencies exhibit the following advantages:
  - Smaller devices
  - Better channels for transmission
  - Lower level of noise and interferences
  - Allowing simultaneous transmission of multiple signals in differente frequencies (FDMA)

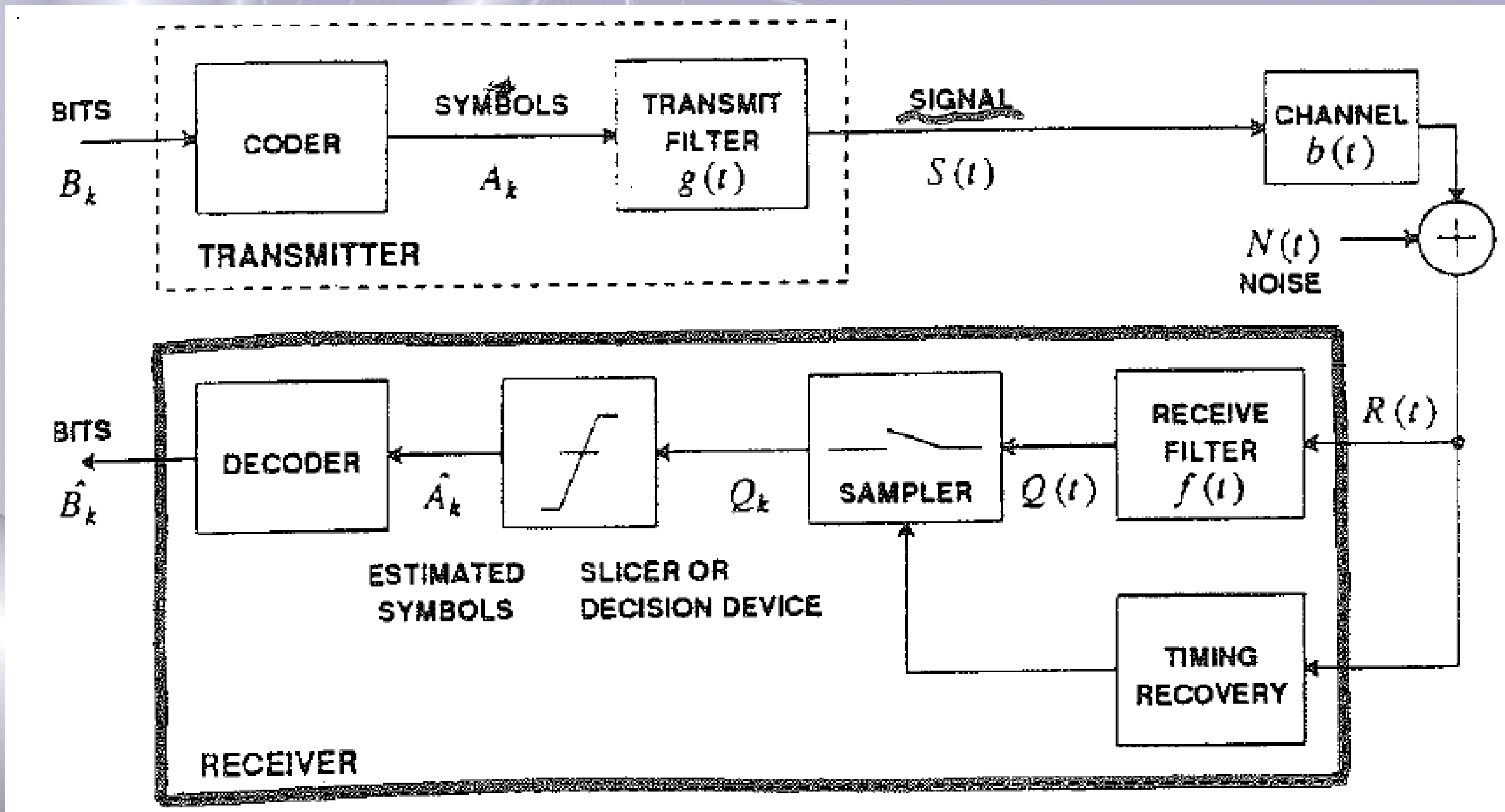


From Baseband to Passband

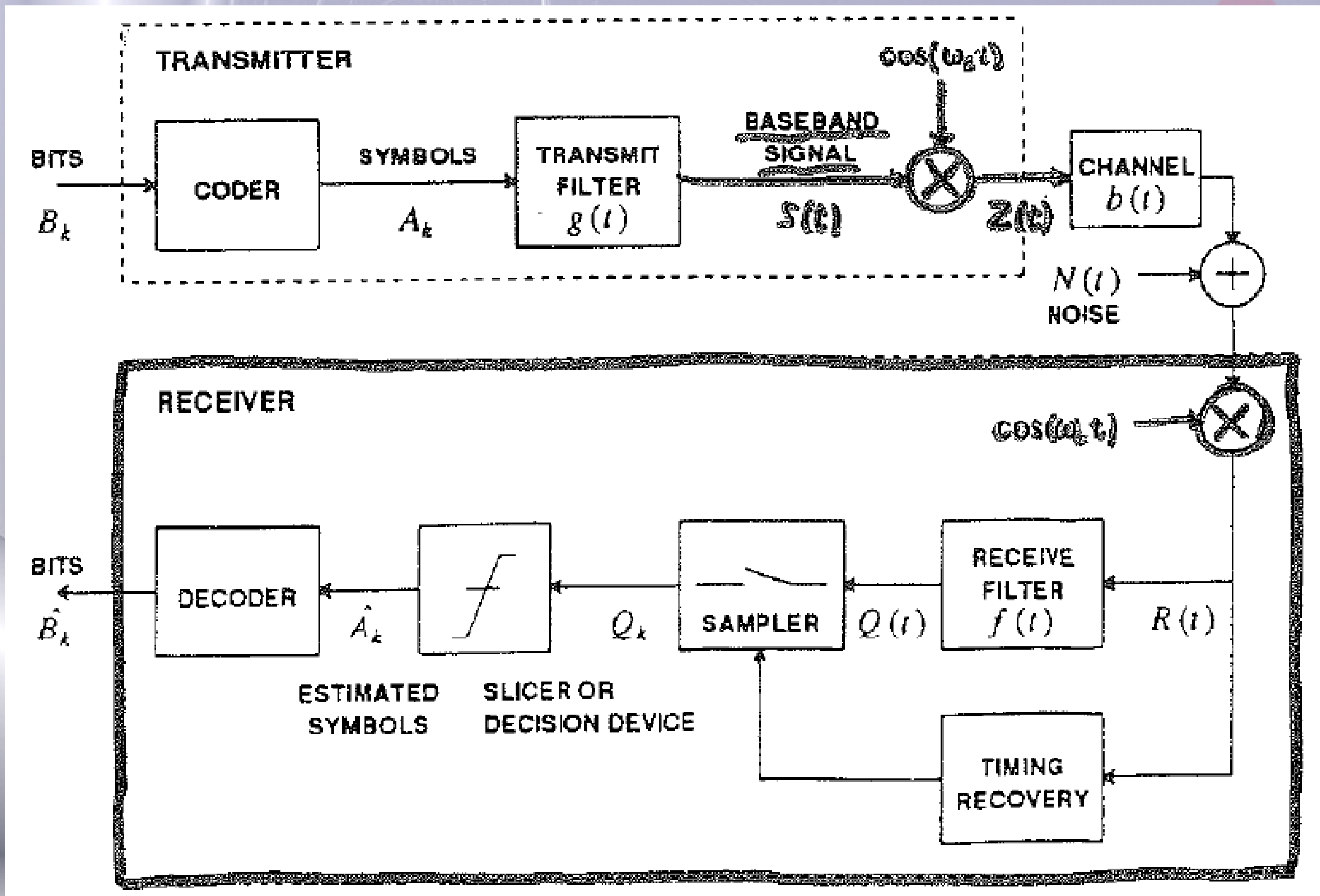
- Baseband singal = original signal without any processing.
  - Spectrum of the signal as “it is”. Bandwidth ocuppied depending of the nature of the signal
- How can we transform the Spectrum of the signal without changing the information contained in it



Example of Digital System working with Baseband singals



Example of Digital System working with Bandpass signals

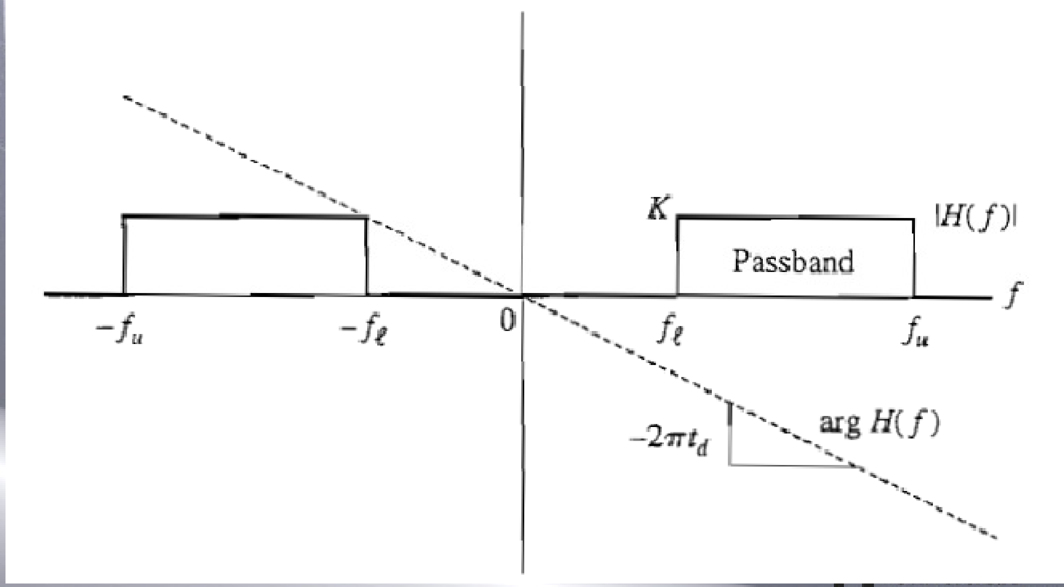


Filtering

- Filtering concept
  - Every transmission system includes one or more “Filters” that contribute to reject all out-of-band undesired signals, clearing the desired signal
  - Filters are defined by their starting and ending passband frequencies, or equivalently their bandwidth and central frequency.
- Ideal filters:

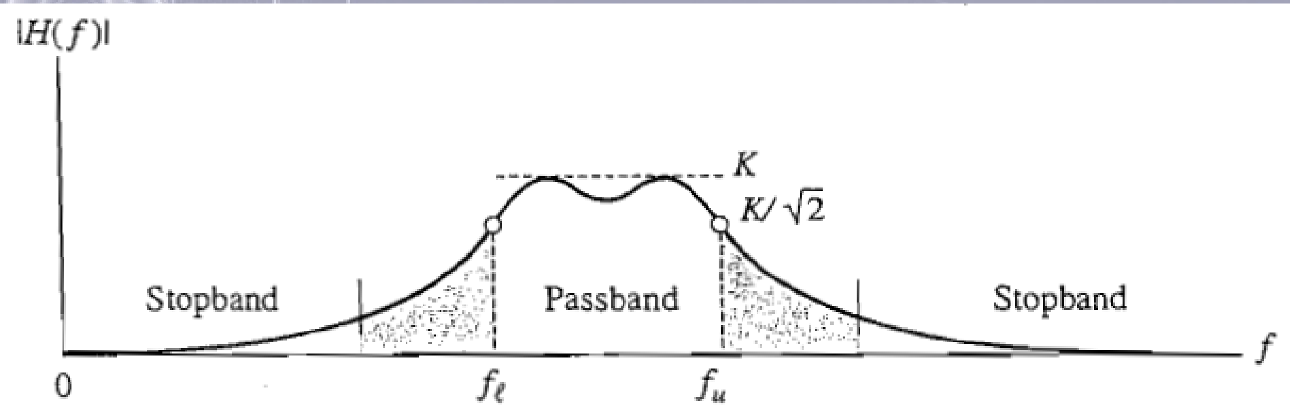
$$H(f) = \begin{cases} Ke^{-j\omega t_d} & f_\ell \le |f| \le f_u \\ 0 & \text{otherwise} \end{cases}$$

$$B = f_u - f_\ell$$



Filtering (II)

- Actual filters
  - Ideal filters can not implemented within finite impulse response. Actual approximations are used to approximate ideal response



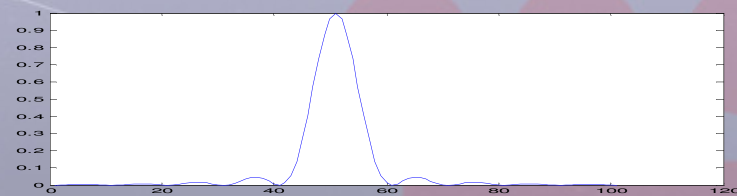
- Filter types:
  - Low-Pass
  - High-Pass
  - Band-Pass
  - Band-Stop

Equivalent Bandwidth for Actual Systems

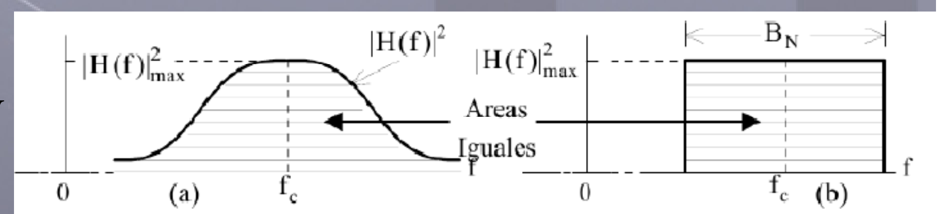
- Example: $x(t) = \Pi(t/T) \Rightarrow X(f) = T \cdot sinc(fT)$

  $T = 10^{-4} seg$

  Is BandWidth (BW) = ∞?



- BW to first null:
  - Ej.: BW = 1/T = 10KHz
- Noise Equivalent BW:
  - Same power than ideal filter of BW
  - Ex.: BW = 1/(2T) = 5KHz



- 3dB BW:
  - $G(f_{3dB}) = G(0) / 2$
  - Ex.: BW = 4.5KHz

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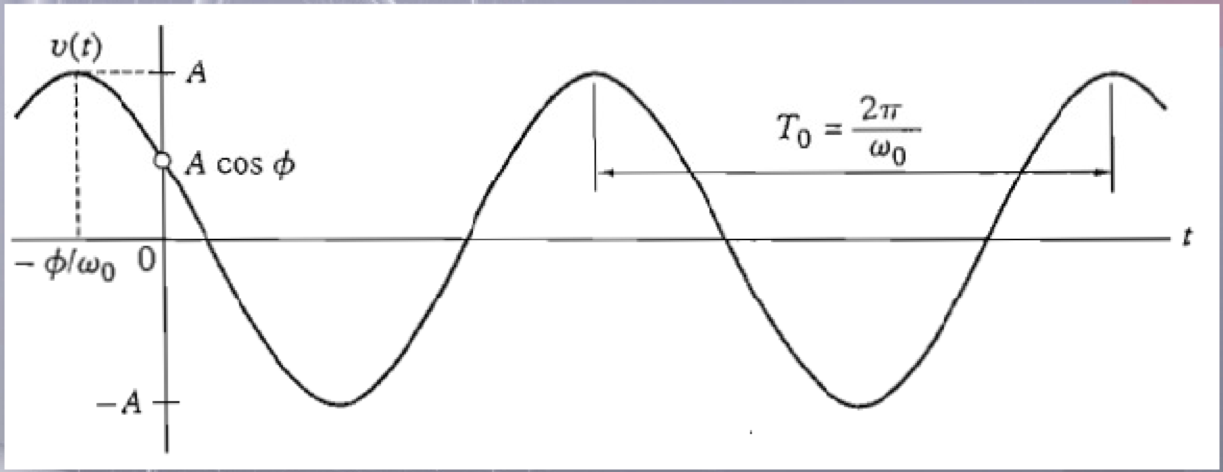
Signal Representation

- We look for simple models to describe signals
- Requirements for the models:
  - It must include all the valueable information of the signal – no information loss
  - It should be as simplest as possible
- Example: to represent a straight line -> duple(point, slope)

Phasors

- Assuming a Sin waveform:
  - We only need to know: amplitude, frequency and phse

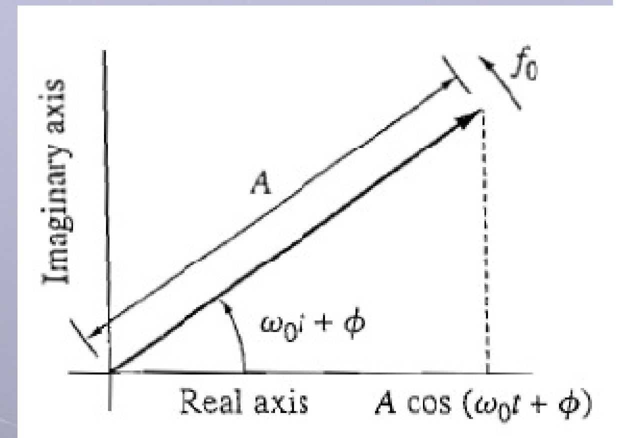
$$v(t) = A \cos(\omega_0 t + \phi)$$



- Recall the Euler's theorem:

$$e^{\pm j\theta} = \cos\theta \pm j \sin\theta$$

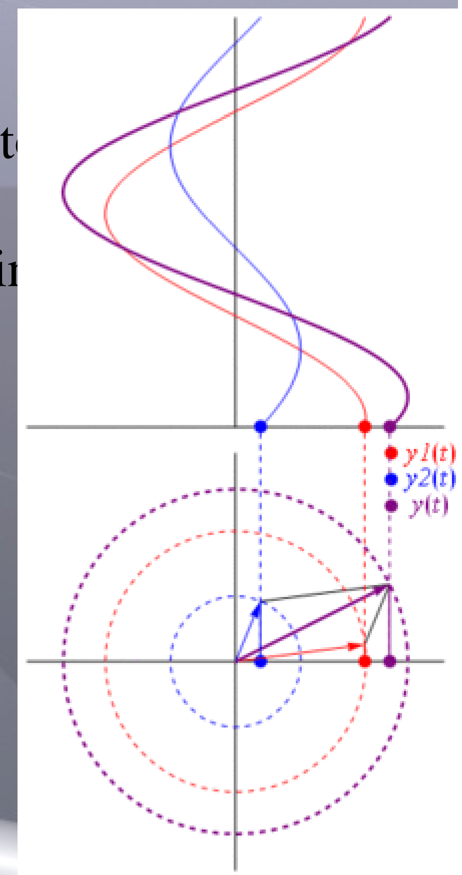
Phasors (II)



- Phasor's description of a tone:

$$A \cos(\omega_0 t + \phi) = A\,\mathrm{Re}\left[e^{j(\omega_0 t + \phi)}\right] = \mathrm{Re}\left[A e^{j\phi} e^{j\omega_0 t}\right]$$

  - The term between brackets can be understood as a vect spinning around
  - The projection of the phasor over the real axis is the Si



- Convections for Phasors:
  - The angle/phase is measured respect the cosine
  - The amplitude is always considered positive

$$\sin \omega t = \cos(\omega t - 90°)$$

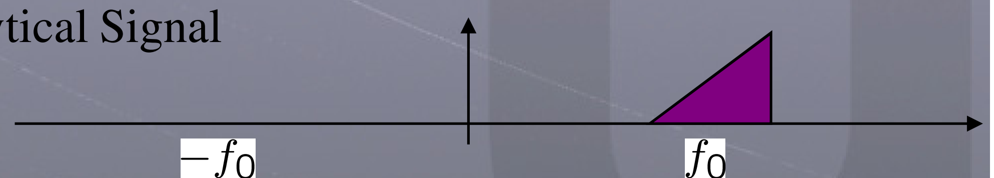
$$-A \cos \omega t = A \cos(\omega t \pm 180°)$$

Low Pass Equivalent: Objetive

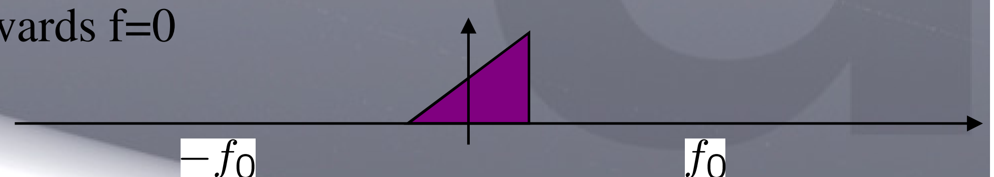
- Low Pass Equivalent intends to be a representation of an actual signal that:
  - Be equivalent -> contains the same information
  - Low Pass -> it Power/Energy Spectral Density is low pass, i.e. centered around f=0

- Graphically
  - We start with the original signal

  - Is there redundant info – Analytical Signal



  - Now, we shift the spectrum towards f=0



Starting Point: Band-Pass Signals

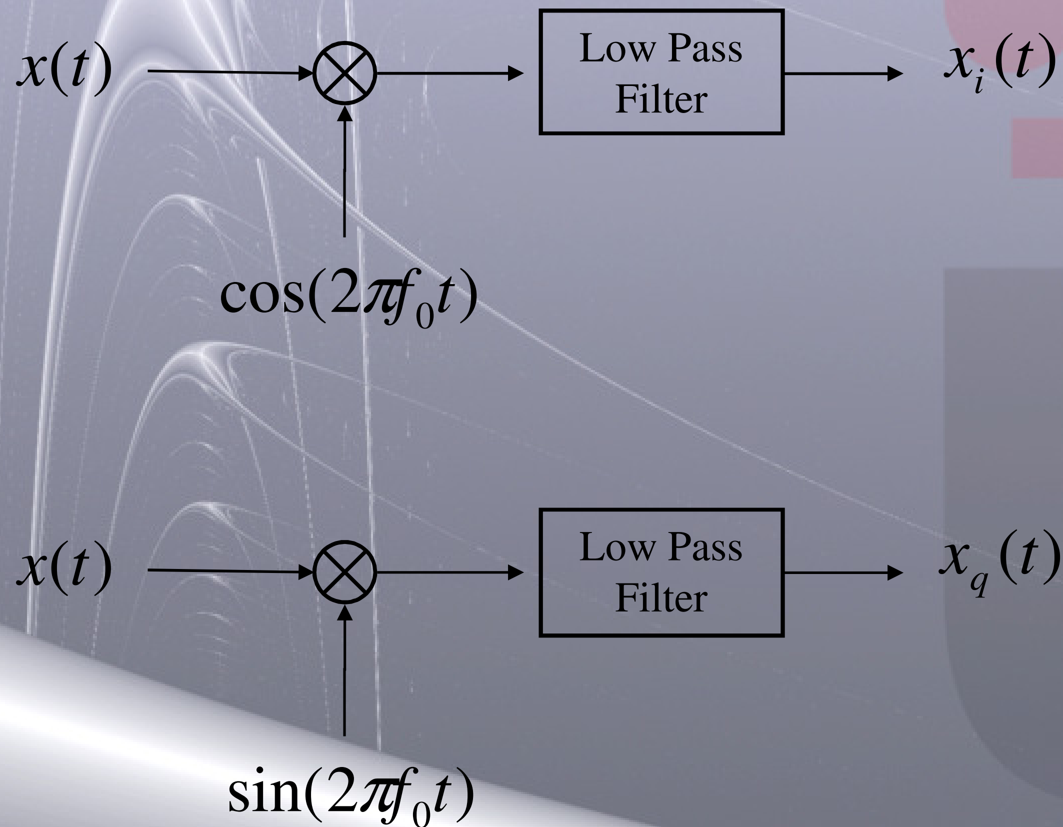
- Any band-pass signal centered around f_0 can be re-writen as:

$$
\begin{aligned}
x(t) &= A_c(t)\cos(2\pi f_0 t + \phi(t)) = \\
&= A_c(t)\cos(\phi(t))\cos(2\pi f_0 t) - A_c(t)\sin(\phi(t))\sin(2\pi f_0 t) = \\
&= x_i(t)\cos(2\pi f_0 t) - x_q(t)\sin(2\pi f_0 t)
\end{aligned}
$$

  - Where we get two components that are orthogonal to each other
- So, we define two components:
  - In-Phase $x_i(t)$: which is the result of projecting *x(t)* over cosine carrier
  - Quadrature $x_q(t)$: projection of *x(t)* over sine carrier
    - Note that for mathematical neatness quadrature component appears with a (-) minus sign

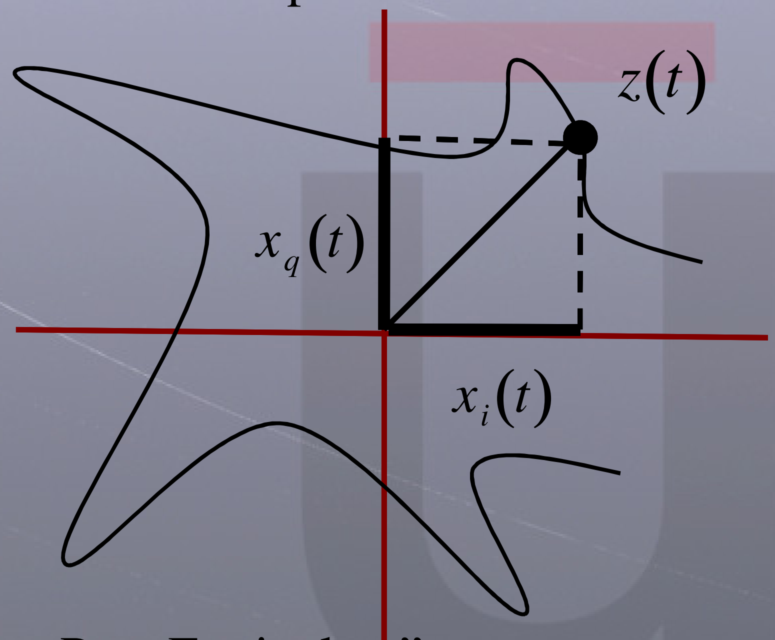
Starting Point: Band-Pass Signals (II)

$$\boxed{x(t) = x_i(t)\cos(2\pi f_0 t) - x_q(t)\sin(2\pi f_0 t)}$$



In-Phase and Quadrature Components

- It can be defined the complex valued signal $z(t)$ as
  $z(t) = x_i(t) + \mathrm{j}\, x_q(t)$
  - Graphically we can think of this signals as two components in a complex plane



  - The complex signal is regarded as "Low Pass Equivalent" or "Complex Envelope"

In-Phase and Quadrature Components (II)

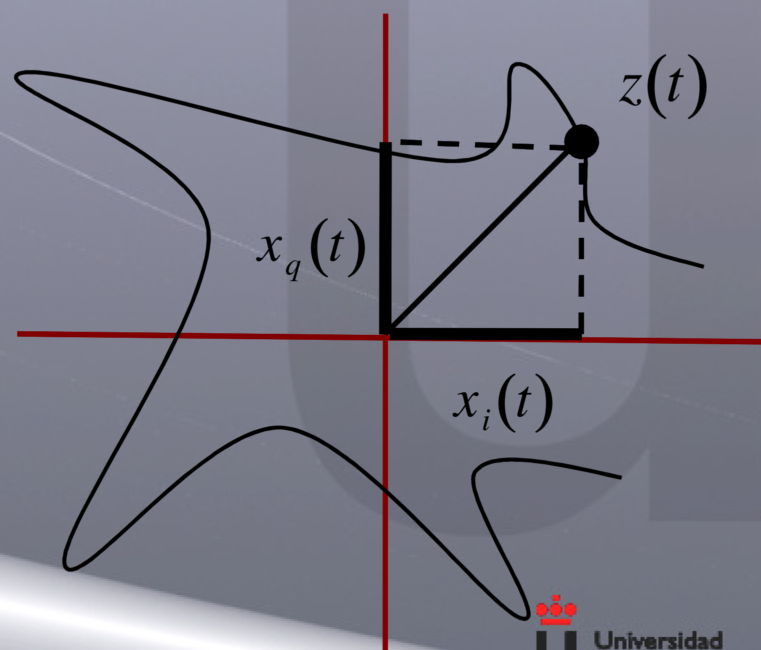
- The “Low-Pass-Equivalent” or “Complex Envelope” approach allows to easily to the carrier **instantaneous envelope/amplitude** and carrier **instantaneous phase**:

$$\phi(t) = \arctan\left(\frac{x_q(t)}{x_i(t)}\right)$$

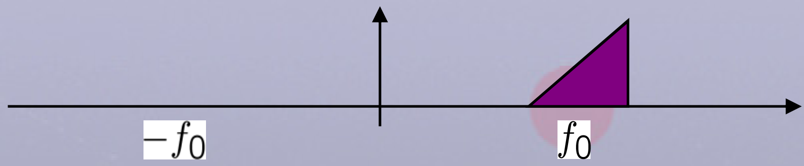
$$A(t) = \sqrt{x_q^2(t) + x_i^2(t)}$$

$$x_i(t) = A(t)\cos(\phi(t))$$

$$x_q(t) = A(t)\sin(\phi(t))$$



Analytical Signal



- The analytical Signal associated to a real-valued signal, $x(t)$, is to satisfy the following properties:
  - Complex valued (➔ two components: real and imaginary axis).
  - Real axis should coincide with real-valued signal , $x(t)$
  - Its spectrum is null for negative values of frequencies

- So, the quadrature component has to be computed, such that:

$$z(t) = x(t) + j\hat{x}(t) = A_z(t) \cdot e^{j\phi_z(t)}$$

  - The quadrature component is to be the Hilbert Transform of the real valued signal

Hilbert Transform

- It can be proven that by defining the quadrature component as
$$\hat{x}(t) = \frac{1}{\pi t} * x(t)$$
all the restrictions on the analytical singal are satisfied
- This signal is obtained as the Hilbert Transform of $x(t)$
$$z(t) = x(t) + j\hat{x}(t) = x(t) + j\frac{1}{\pi t} * x(t) = x(t) * \left(\delta(t) + j\frac{1}{\pi t} \right)$$
  - In other words, from a real valued singal, a different real valued signal can be obtained using the Hilbert Transform and combination of both signals gives the analytical signal

Hilbert Transform (II)

- Consequently, Hilbert Transform is defined as:

$$\hat{x}(t) = \frac{1}{\pi t} * x(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} x(\tau) \frac{1}{t-\tau} d\tau$$

and therefore, in frequency domain, it is defined through the following frequency response:

$$TF\left\{\frac{1}{\pi t}\right\} = \begin{cases} -j, & f > 0 \\ 0, & f = 0 \\ j, & f < 0 \end{cases} = \begin{cases} e^{-j\pi/2}, & f > 0 \\ 0, & f = 0 \\ e^{j\pi/2}, & f < 0 \end{cases} = -j \cdot sign(f)$$

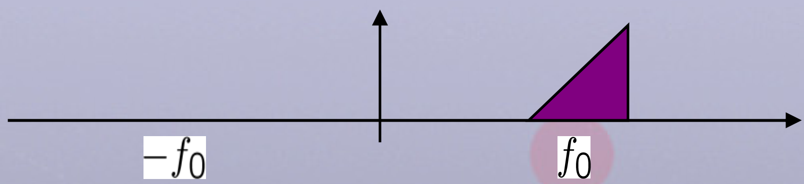
$$\hat{X}(f) = [-j \cdot sign(f)] X(f)$$

Hilbert Transform (III)

- Properties of the HT:
  - It changes phase of signal, but it does not alter its module
    $$\left| -j \cdot sign(f) \right| = 1$$
  - Any signal and its HT are orthogonal
    $$x(t) \perp \hat{x}(t)$$
  - $\hat{\hat{x}}(t) = -x(t)$
  - If $x(t)$ is a band-pass signal and $m(t)$ a low-pass one, and their spectrums do not overlap
    $$c(t) = m(t)x(t) \Rightarrow \hat{c}(t) = m(t)\hat{x}(t)$$
- Well known HT pairs

$$x(t) = \cos(2\pi \cdot f_0 t) \Rightarrow \hat{x}(t) = \sin(2\pi \cdot f_0 t)$$
$$x(t) = \sin(2\pi \cdot f_0 t) \Rightarrow \hat{x}(t) = -\cos(2\pi \cdot f_0 t)$$

Analytical Signal (II)



- Given a real valued signal, $x(t)$, which Power/Energy Spectral Density is defined by $S_x(f)$, the corresponding Anaylytical Signal has an spectrum defined by:

$$S_z(f) = 2S_x(f)U(f)$$

where $U(f)$ is the Fourier Transform of:

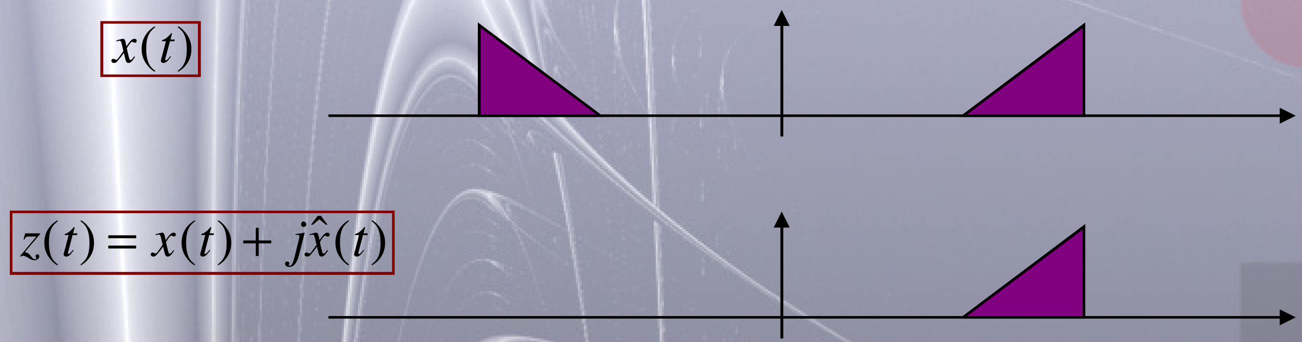
$$U(f) = TF\left\{\delta(t) + j\frac{1}{\pi t}\right\} = \begin{cases} 1 & f > 0 \\ 0, & \text{elsewhere} \end{cases}$$

satisfying:

$$U(f) = \left|U(f)\right|^2$$

Analytical Signal (III)

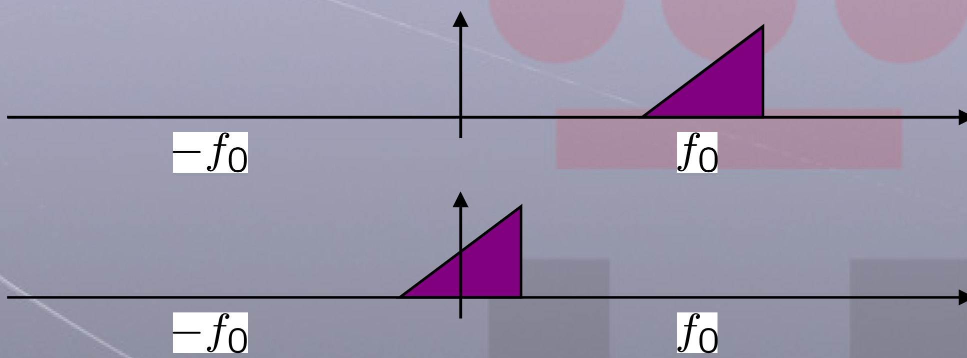
- Intuitively, the plot of the Power Spectral Density are:



  - Analytical signal and HT have been proposed on previous slides for real valued signal. However they can be generalized for complex signals.
    - Generaly speaking, HT provides a 90º phase shifted version of the original signal

Low-Pass Equivalent Signal

- From the Analytical Signal, what is missing to get the Low-Pass Equivalent Signal?



  - Frequency shift by f_0 to the left
  - In frequency domain that is to convolve the spectrum with $\delta(f - f_0)$

    or, equivalently, to multiply by $e^{j2\pi f_0 t}$ in the time domain

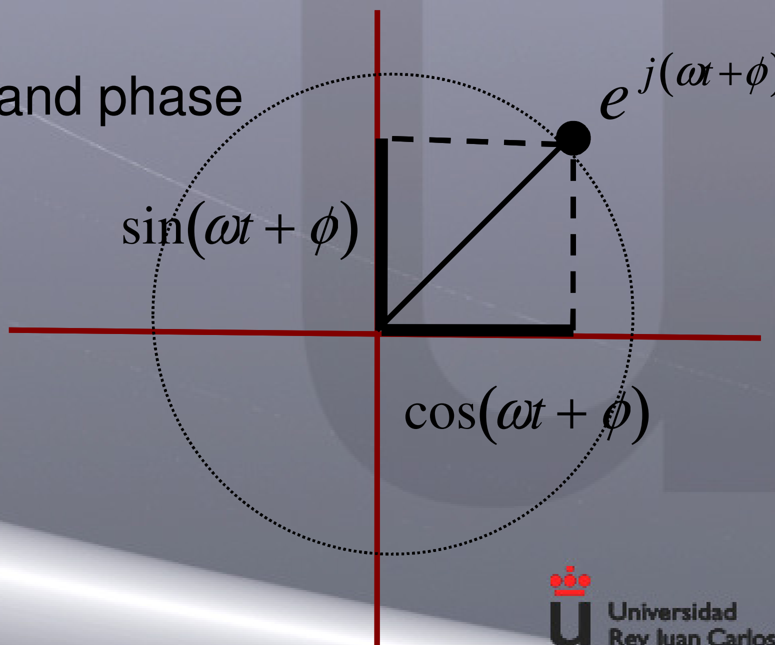
Recall of Complex Exponential Signal

- Euler's rule:

$$e^{j(\omega t+\phi)} = \cos(\omega t + \phi) + j\sin(\omega t + \phi)$$

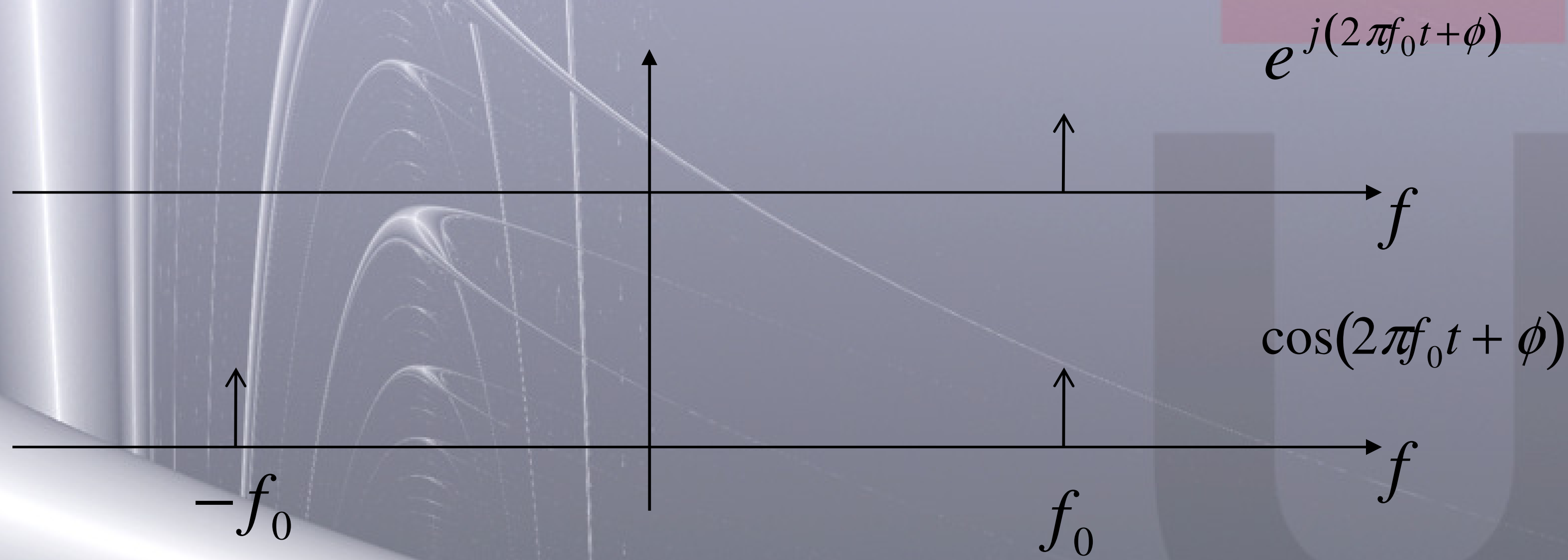
  – So, cosine and sine can be seen as the in-phase and quadrature components of the complex exponential

  – Neat separation between amplitude and phase



Recall of Complex Exponential Signal (II)

- The Power Spectral Density of the Complex Exponential signal has only a positive frequency contribution



Recall of Complex Exponential Signal (III)

- Summarizing:
  - A real valued sine signal can be seen as one of the components of a complex exponential signal
  - A complex exponential is the natural extension of sine and cosine waves into the complex plane
  - Complex exponentials allows us to easily separate phase and amplitude in communication signals
  - Positive Spectrum of a sine (pass-band) signal and complex exponential (analytical) match each other.
  - Complex exponential (analytical) has not power at negative frequencies, while sine signals (real valued) do

Low-Pass Equivalent

- Band-Pass (Real Valued)

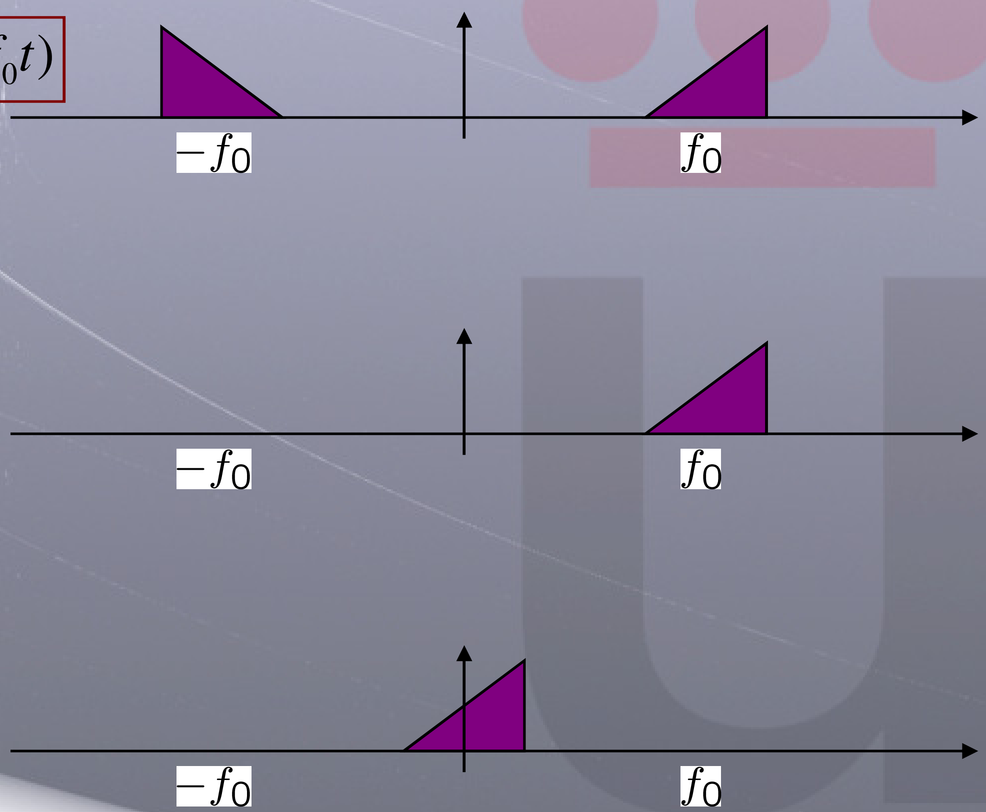
$$x(t) = x_i(t)\cos(2\pi f_0 t) - x_q(t)\sin(2\pi f_0 t)$$

- Analytical Signal

$$z(t) = x(t) + j\hat{x}(t) = \\ = x(t) * \left(\delta(t) + \frac{j}{\pi t}\right)$$

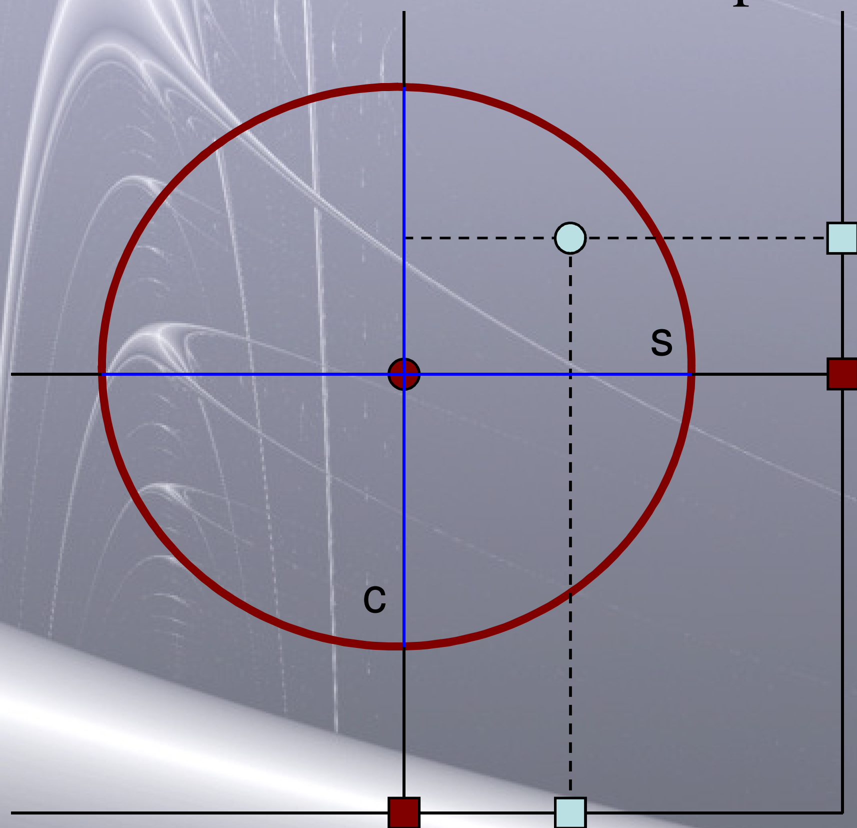
- Low-Pass Equivalent

$$x_l(t) = x_i(t) + jx_q(t) = \\ = z(t)e^{-j2\pi f_o t}$$



In-Phase and Quadrature Modulation

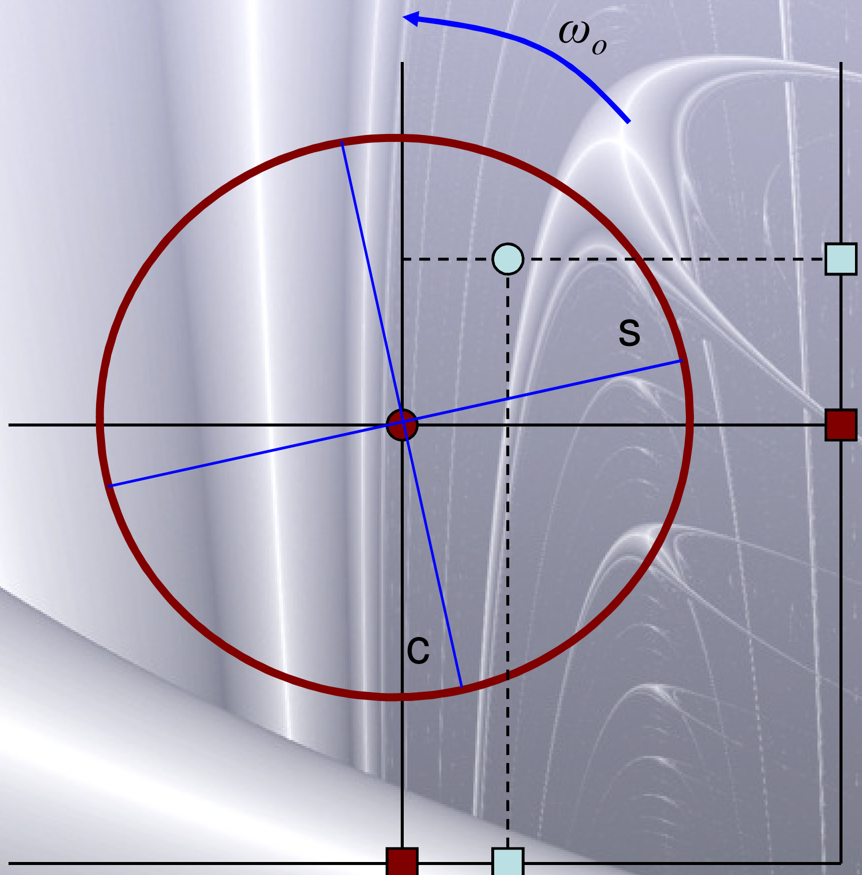
- In a Band-Pass modulation, both components (In-Phase and Quadrature) can be modulated with independent information



$$x_q(t) = m_2(t)$$

$$x_i(t) = m_1(t)$$

In-Phase and Quadrature Modulation (II)



$$\hat{x}(t) = m_2(t)\cos(2\pi f_0 t) + m_1(t)\sin(2\pi f_0 t)$$

- The transmitted signal, *x(t)*
  - Is band-pass
  - Is real-valued
  - Transmits double information while it occupies the same bandwidth

$$x(t) = m_1(t)\cos(2\pi f_0 t) - m_2(t)\sin(2\pi f_0 t)$$

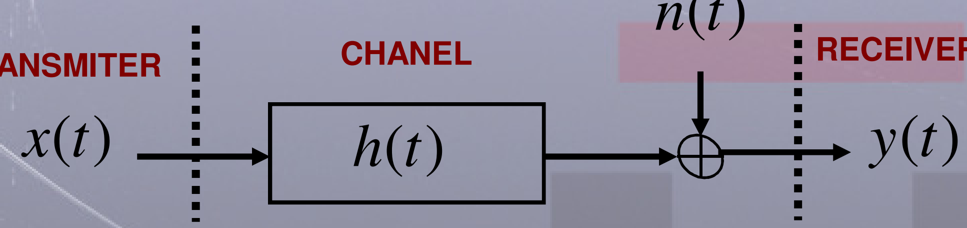
Channel's Model

- We know that transmitted signals arrive to the receiver suffering some kinds of degradation
  - Attenuation $$y(t) = \frac{x(t)}{L}$$
  - Delay $$y(t) = x(t-\tau)$$
  - Noise
    - From different sources
    - Sometimes it can be modeled as Gaussian $$y(t) = x(t) + n(t)$$
  - Time dispersion
    - General concept of linear convolution $$y(t) = x(t) * h(t)$$

Channel's Model

- We have a general model for the channel :

$$y(t) = x(t) * h(t) + n(t)$$

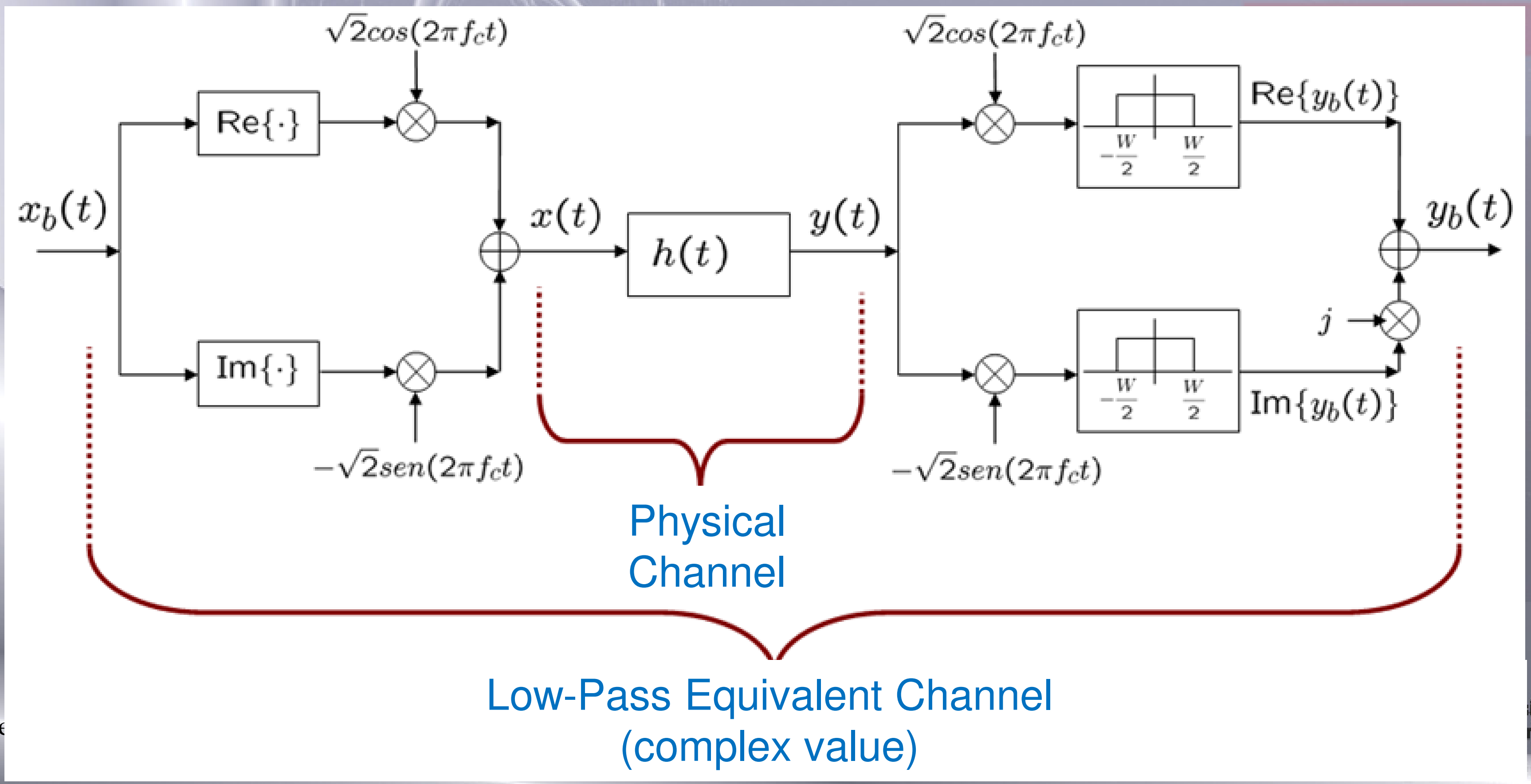


- We can further simplify it by:
  - Working with Base Band Equivalent
  - Working on Discrete Time Equivalent by sampling signals and channel response, T_s.

➔ We get the Low-Pass Discrete Equivalent model

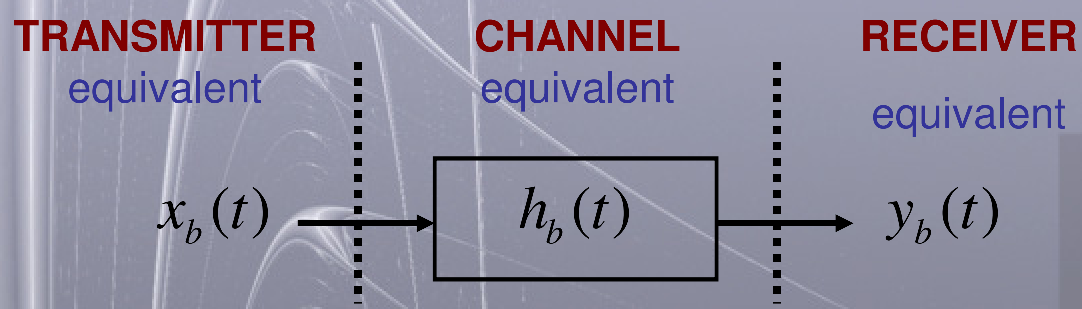
Low-Pass Discrete Equivalent Channel

- Assuming the signal and the channel response are band-pass
  - Signals can be expressed by their Low-Pass Equivalent



Low-Pass Discrete Equivalent Channel

- A Low-Pass Equivalent model for the overall transmitter-channel-receiver can be defined as in previous slide

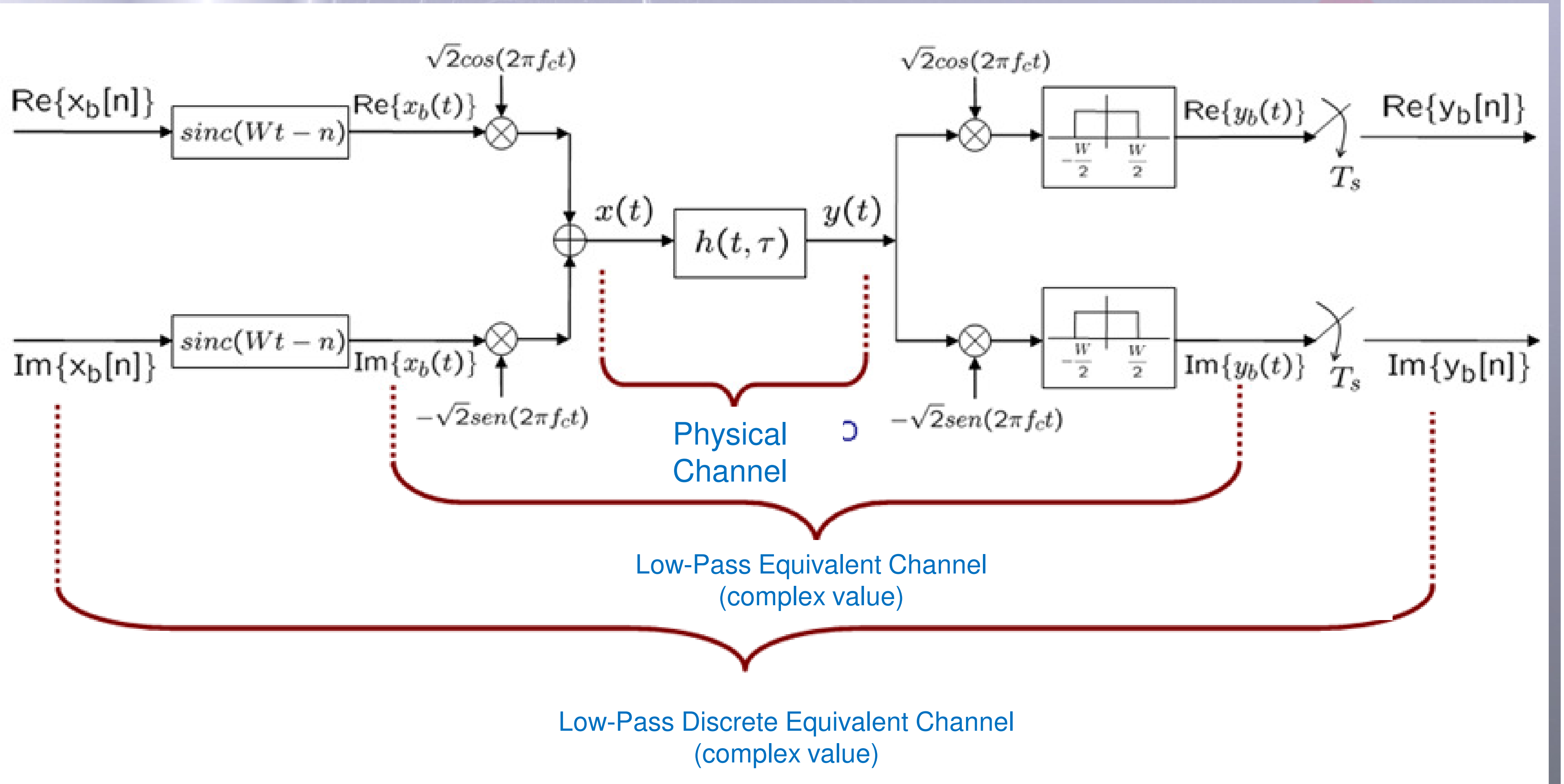


  - Where $h_b(t)$ is the low-pass equivalent of h(t)

– All these signals are band-limited, therefore they can be sampled following the Nyquist rule and there is not loss of information

$$x_b(t) = \sum_n x_b[n] \operatorname{sinc}(W_s t - n)$$

Low-Pass Discrete Equivalent Channel



Low-Pass Discrete Equivalent Channel

- A model for the noise has to be added too
  - The signals are complex and discrete ➔ noise should be complex and discrete, with the same characteristics than a AWGN sampled process

$$w[n] \rightarrow G(0, N_0/2) + j \cdot G(0, N_0/2) \rightarrow G_{Comp}(0, N_0)$$

- So, the Low-Pass Discrete Equivalent signal at the receiver is

$$y_b[n] = h_b * x_b[n] + w[n]$$

Summary of Concepts in this Chapter

- Any Band-Pass signal can be decomposed into its In-Phase and Quadrature components
  - So, any Band-Pass signal can be decomposed into two orthogonal signals
  - That can be easily separated at the receiver
  - And they can be modulated independently (different info)
- The Low-Pass Equivalent of a real value signal is a complex signal:
  - Low-Pass equivalent does not provide information about the carrier frequency. It is supposed to be known
- All the above signal representation are of great value when analyzing communication signals and systems.
- A whole communications link can be analyzed using their Low-Pass Discrete Equivalent models for the transmitted signal, the channel, noise and for receiver.