WORKSHEET 1: The real line. Functions

1. Solve the following inequalities, determine the set of real numbers that satisfies them. Represent its solution using a real line.

a)(*) $|9-2x|<1$ b)(*) $-5|x+3|<4x-5$ c)(*) $\dfrac{|x|}{3}+2<|x|$

d)(*) $1<|3-2x|$ e)(*) $\dfrac{(x^2-16)(x-1)}{x-3}\geq 0$ f) $|x-3|+|x+3|<10$

a) $Sol: (4,5)$

b) $Sol: (-\infty,-20)\cup(-10/9,\infty)$.

c) $Sol: (-\infty,-3)\cup(3,\infty)$.

d) $Sol: (-\infty,1)\cup(2,\infty)$.

e)$Sol: (-\infty,-4]\cup[1,3)\cup[4,\infty)$.

f) $Sol.: (-5,5)$.

2. (*)Interpret the previous inequalities a), b), c) and d) geometrically using functions.

a) $y=|9-2x|$; $y=1$ b) $y=-5|x+3|$; $y=4x-5$

c) $y=\dfrac{|x|}{3}+2$; $y=|x|$ d) $y=1$; $y=|3-2x|$

3. Discuss if the following inequalities are satisfied:

a) (*) $|x+y|\leq|x|+|y|$ b)(*) $|x|+|y|\leq|x+y|$ c)(*) $|x-y|\leq|x|+|y|$ d)(*) $|x|-|y|\leq|x-y|$

a) Always true.

b) True only if $x,y\in[0,\infty)$ or if $x,y\in(-\infty,0]$.

c) Always true.

d) Always true.

4. Discuss if the following implications are true or false.

a) $x<y\Rightarrow x^2<y^2$ b) $|x|<|y|\Rightarrow x^2<y^2$

c) $x^2<y^2\Rightarrow x<y$ d) $x^2<y^2\Rightarrow|x|<|y|$

a) If $y\leq 0$, always false; if $0\leq x$,always true; rest of the examples, it depends on.

b) and d) always true.

c) If $y<0$, always false; if $0<y$,always true; if $y=0$,it is impossible.

5. Obtain for the following sets of real numbers, $A\subset\mathbb{R}$, their maximum and minimum elements whenever they exist. Calculate them for $\alpha=-1$, $\alpha=0$ and $\alpha=1$

a) $A=\{x: e^x\leq\alpha\}$ b) $A=\{x: e^x\geq\alpha\}$ c) $A=\{x:\ln x\leq\alpha\}$ d) $A=\{x:\ln x\geq\alpha\}$

a) if $\alpha=-1$ or if $\alpha=0\Longrightarrow A$ has no maximum nor minimum; if $\alpha=1\Longrightarrow A$ has no minimum, but max(A)=0.

b) if $\alpha=-1$ or if $\alpha=0\Longrightarrow A$ has no maximum nor minimum; if $\alpha=1\Longrightarrow A$ has no maximum, but min(A)=0.

c) if $\alpha=-1,\alpha=0$ or if $\alpha=1\Longrightarrow A$ has no minimum, but max(A)=$e^{-1},1,e$, respectively.

d) if $\alpha=-1,\alpha=0$ or if $\alpha=1\Longrightarrow A$ has no maximum, but min(A)=$e^{-1},1,e$, respectively.

6. (*)Given the functions $f(x)=1/x$ and $g(x)=x^2-1$.

a) Find their domain and range.
b) Calculate $f(g(2))$ and $g(f(2))$.
c) Obtain the functions $f(g(x))$ and $g(f(x))$.

a) $Dom(f)=(-\infty,0)\cup(0,\infty)=Im(f)$. $Dom(g)=\mathbb{R}, Im(g)=[-1,\infty)$.

b) $f(g(2))=\frac{1}{3}$ and $g(f(x))=-\frac{3}{4}$.

c) $f(g(x))=\frac{1}{x^2-1}$ and $g(f(x))=\frac{1}{x^2}-1$.

7. Review the graphs of the following functions: a)(*) $f(x) = x^2$ b)(*)$f(x) = e^x$ c)(*)$f(x) = lnx$

In each case sketch the graph transformations of the previous ones, interpreting geometrically the results.

i) $g(x) = f(x+1)$ ii) $h(x) = -2f(x)$ iii) $p(x) = f(3x)$
iv) $s(x) = f(x) + 1$ v) $r(x) = |f(x)|$ vi) $m(x) = f(|x|)$

i) Translate the graph one unit to the left.

ii) Stretch the graph vertically ($2f(x)$) and, then, make a reflection with respect to the horizontal axis ($-2f(x)$).

iii) Compress horizontally the graph.

iv) Translate vertically the graph one unit upwards.

v) Keep invariant the part of the graph that stays above the horizontal axis, and obtain the symmetric part with respect to the horizontal axis, for the part of the graph that stays below the horizontal axis.

vi) Keep invariant the part of the graph that stays at the right of the vertical axis, suppress the part of the graph that stays at the left of the vertical axis and substitute it for the symmetric part of the graph that remains at the right of the vertical axis.

8. (*)Find the domain and the range of the following functions:

a)$f(x) = ln(1 + |x|)$ b)$g(x) = 2 - \sqrt{1 - x^2}$ c) $h(x) = e^{\sqrt{1-x^2}}$

a) $Dom(f) = \mathbb{R}, Im(f) = [0, \infty)$

b) $Dom(g) = [-1, 1], Im(g) = [1, 2]$

c) $Dom(h) = [-1, 1], Im(h) = [1, e]$

9. (*)Let $f,\, g : I \to \mathbb{R}$ be two increasing functions. Discuss if the following statements are true or false.

a) $f + g : I \to \mathbb{R}$ is an increasing function.

b) $f \cdot g : I \to \mathbb{R}$ is an increasing function.

a) Obviously.

b) If f, g positive: obviously.

10. Let $f,\ g : \mathbb{R} \to \mathbb{R}$ be monotonic functions. Discuss if the composition $g \circ f$ is increasing or decreasing depending on the monotonicity of f and g. (There are four diferent cases)

a) gof will be increasing if both are increasing or decreasing.

b) gof will be decreasing if one of them is increasing and the other one is decreasing.

11. Find the intervals I, J for the following functions so that they are bijective. For example ($f : I \to J$)

a) $f(x) = x^2$ b) $g(x) = \ln|x|$ c) $h(x) = e^{-x^2}$

a) $f : (-\infty, 0] \longrightarrow [0, \infty)$, $f : [0, \infty) \longrightarrow [0, \infty)$

b) $g : (-\infty, 0] \longrightarrow \mathbb{R}$, $f : [0, \infty) \longrightarrow \mathbb{R}$

c) $h : (-\infty, 0] \to (0, 1]$, $h : [0, \infty) \to (0, 1]$.

12. Determine if the following functions are even, odd or neither:

a) $f(x) = \dfrac{x^2}{x^2 + 1}$ b) $g(x) = \dfrac{x^3}{x^4 + 1}$ c) $h(x) = \dfrac{x^3}{x^5 + 1}$

a) Even; b) Odd; c) not even nor odd.

13. (*)Calculate the inverse function:

$f(x) = (x^3 - 5)^5, \quad g(x) = (\sqrt[3]{x} - 5)^5$

a) $f^{-1}(x) = \sqrt[3]{5 + \sqrt[5]{x}}$.

b) $g^{-1}(x) = 5 + x^{3/5} = 5 + (\sqrt[5]{x})^3$.

14. Let f be an even function and g an odd one. Prove that:

$|g|$ is even; $f \circ g$ is even; $g \circ f$ is even;
$f \cdot g$ is odd; g^k is even (if k is even); g^k is odd (if k is odd)