Logic

Interesting proofs with Natural Deduction

Damiano Zanardini

Graduado/a en Ingeniería Informática
School of Computer Science
Technical University of Madrid
damiano.zanardini@gmail.com
damiano@fi.upm.es

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To start with

Observations

- all this proofs are obtained
  - either only using basic rules
  - or using previous proofs
- i.e., we never use something unless it has been previously proved or is a basic rule
- equivalencies are proved in two steps: $\rightarrow$ and $\leftarrow$
- after proving a derived rule or one direction of an equivalence (e.g. "DM1" o "cut1") we can use it as a normel rule by calling it by its name
  - we use "cut2" in line 13 of the proof of "cut3"
  - we use "DM1" in line 2 of the proof of "DM2"
- instead of "cut1" and "cut2" we often use simply "cut"; we do the same in similar cases

Derived rules

transitivity: $T[A \to B, \quad B \to C] \vdash A \to C$

1:	A	assumption
2:	$A \to B$	premise
3:	B	(1,2)
4:	$B \to C$	premise
5:	C	(3,4)
6:	$A \to C$	(1-5)

Derived rules

modusTollens : $T[A \rightarrow B, \quad \neg B] \vdash \neg A$

1:	A	assumption
2:	$A \rightarrow B$	premise
3:	B	(1,2)
4:	$\neg B$	premise
5:	$B \wedge \neg B$	(3,4)
6:	$A \rightarrow B \wedge \neg B$	(1-5)
7:	$\neg A$	(6)

Derived rules

cut1: $T[A \vee B, \quad \neg A] \vdash B$

1:	$A \vee B$	premise
2:	$\neg A$	premise
3:	$\quad A$	assumption
4:	$\quad\quad \neg B$	assumption
5:	$\quad\quad A \wedge \neg A$	(2,3)
6:	$\quad \neg B \rightarrow A \wedge \neg A$	(4-5)
7:	$\quad \neg\neg B$	(6)
8:	$\quad B$	(7)
9:	$A \rightarrow B$	(3-8)
10:	$\quad B$	assumption
11:	$B \rightarrow B$	(10)
12:	B	(1,9,11)

The second cut rule, "cut2" is very similar

Derived rules

cut3: $T[A \vee B, \quad C \vee \neg A] \vdash B \vee C$

1:	$\neg(B \vee C)$	assumption
2:	B	assumption
3:	$B \vee C$	(2)
4:	$(B \vee C) \wedge \neg(B \vee C)$	(1,3)
5:	$B \rightarrow (B \vee C) \wedge \neg(B \vee C)$	(2-4)
6:	$\neg B$	(5)
7:	C	assumption
8:	$B \vee C$	(7)
9:	$(B \vee C) \wedge \neg(B \vee C)$	(1,8)
10:	$C \rightarrow (B \vee C) \wedge \neg(B \vee C)$	(7-9)
11:	$\neg C$	(10)
12:	$A \vee B$	premise
13:	A	cut2(6,12)
14:	$C \vee \neg A$	premise
15:	$\neg A$	cut1(11,14)
16:	$A \wedge \neg A$	(13,15)
17:	$\neg(B \vee C) \rightarrow A \wedge \neg A$	(1-16)
18:	$\neg\neg(B \vee C)$	(17)
19:	$B \vee C$	(18)

Derived rules

quodlibet: $T[A \wedge \neg A] \vdash B$

1:	$\neg B$	assumption(arbitrary)
2:	$A \wedge \neg A$	premise
3:	$\neg B \rightarrow A \wedge \neg A$	(1-2)
4:	$\neg\neg B$	(3)
5:	B	(4)

Tautology

tau: $T \vdash A \vee \neg A$

1:	$\neg(A \vee \neg A)$	assumption
2:	A	assumption
3:	$A \vee \neg A$	(2)
4:	$(A \vee \neg A) \wedge \neg(A \vee \neg A)$	(1,3)
5:	$A \rightarrow (A \vee \neg A) \wedge \neg(A \vee \neg A)$	(2-4)
6:	$\neg A$	(5)
7:	$\neg A$	assumption
8:	$A \vee \neg A$	(7)
9:	$(A \vee \neg A) \wedge \neg(A \vee \neg A)$	(1,8)
10:	$\neg A \rightarrow (A \vee \neg A) \wedge \neg(A \vee \neg A)$	(7-9)
11:	$\neg\neg A$	(10)
12:	A	(11)
13:	$A \wedge \neg A$	(5,12)
14:	$\neg(A \vee \neg A) \rightarrow A \wedge \neg A$	(1-13)
15:	$\neg\neg(A \vee \neg A)$	(14)
16:	$A \vee \neg A$	(15)

De Morgan

DM1: $T[\neg(A \vee B)] \vdash \neg A \wedge \neg B$

1:	A	assumption
2:	$A \vee B$	(1)
3:	$\neg(A \vee B)$	premise
4:	$(A \vee B) \wedge \neg(A \vee B)$	(2,3)
5:	$A \rightarrow (A \vee B) \wedge \neg(A \vee B)$	(1-4)
6:	$\neg A$	(5)
7:	B	assumption
8:	$A \vee B$	(1)
9:	$\neg(A \vee B)$	premise
10:	$(A \vee B) \wedge \neg(A \vee B)$	(8,9)
11:	$B \rightarrow (A \vee B) \wedge \neg(A \vee B)$	(7-10)
12:	$\neg B$	(5)
13:	$\neg A \wedge \neg B$	(6,12)

De Morgan

DM2: $T[\neg(A \wedge B)] \vdash \neg A \vee \neg B$

1:	$\neg(\neg A \vee \neg B)$	assumption
2:	$\neg\neg A \wedge \neg\neg B$	DM1(1)
3:	$\neg\neg A$	(2)
4:	A	(3)
5:	$\neg\neg B$	(2)
6:	B	(5)
7:	$A \wedge B$	(4,6)
8:	$\neg(A \wedge B)$	premise
9:	$(A \wedge B) \wedge \neg(A \wedge B)$	(7,8)
10:	$\neg(\neg A \vee \neg B) \rightarrow (A \wedge B) \wedge \neg(A \wedge B)$	(1-9)
11:	$\neg\neg(\neg A \vee \neg B)$	(10)
12:	$\neg A \vee \neg B$	(11)

De Morgan

DM3: $T[\neg A \wedge \neg B] \vdash \neg(A \vee B)$

1:	$A \vee B$	assumption
2:	$\neg A \wedge \neg B$	premise
3:	$\neg A$	(2)
4:	$\quad A$	assumption
5:	$\qquad \neg B$	assumption
6:	$\qquad A \wedge \neg A$	(3,4)
7:	$\quad \neg B \rightarrow A \wedge \neg A$	(5-6)
8:	$\quad \neg\neg B$	(7)
9:	$\quad B$	(8)
10:	$A \rightarrow B$	(4-9)
11:	$\quad B$	assumption
12:	$B \rightarrow B$	(11-11)
13:	B	(1,10,12)
14:	$\neg B$	(2)
15:	$B \wedge \neg B$	(13,14)
16:	$(A \vee B) \rightarrow B \wedge \neg B$	(1-15)
17:	$\neg(A \vee B)$	(16)

De Morgan

DM4: $T[\neg A \vee \neg B] \vdash \neg(A \wedge B)$

1:	$A \wedge B$	assumption
2:	A	(1)
3:	$\quad \neg A$	assumption
4:	$\quad A \wedge \neg A$	(2,3)
5:	$\neg A \rightarrow A \wedge \neg A$	(3-4)
6:	$\neg\neg A$	(5)
7:	B	(1)
8:	$\quad \neg B$	assumption
9:	$\quad B \wedge \neg B$	(7,8)
10:	$\neg B \rightarrow B \wedge \neg B$	(8-9)
11:	$\neg\neg B$	(10)
12:	$\neg\neg A \wedge \neg\neg B$	(6,11)
13:	$\neg(\neg A \vee \neg B)$	DM3(12)
14:	$\neg A \vee \neg B$	premise
15:	$(\neg A \vee \neg B) \wedge \neg(\neg A \vee \neg B)$	(13,14)
16:	$(A \wedge B) \rightarrow (\neg A \vee \neg B) \wedge \neg(\neg A \vee \neg B)$	(1-15)
17:	$\neg(A \wedge B)$	(16)

Associativity

$\wedge$-associativity: $T[(A \wedge B) \wedge C] \vdash A \wedge (B \wedge C)$

1:	$(A \wedge B) \wedge C$	premise
2:	$A \wedge B$	(1)
3:	A	(2)
4:	B	(2)
5:	C	(1)
6:	$B \wedge C$	(4,5)
7:	$A \wedge (B \wedge C)$	(3,6)

The other direction is very similar

Associativity

$\vee$-associativity: $T[(A \vee B) \vee C] \vdash A \vee (B \vee C)$

1:	$\neg(A \vee (B \vee C))$	assumption
2:	$\neg A \wedge \neg(B \vee C)$	DM1(1)
3:	$\neg A \wedge (\neg B \wedge \neg C)$	int(2) with DM1
4:	$(\neg A \wedge \neg B) \wedge \neg C$	$\wedge$-asociatividad
5:	$\neg(A \vee B) \wedge \neg C$	int(4) with DM3
6:	$\neg((A \vee B) \vee C)$	DM3(5)
7:	$(A \vee B) \vee C$	premise
8:	$((A \vee B) \vee C) \wedge \neg((A \vee B) \vee C)$	(6,7)
9:	$\neg(A \vee (B \vee C)) \to ((A \vee B) \vee C) \wedge \neg((A \vee B) \vee C)$	(1-8)
10:	$\neg\neg(A \vee (B \vee C))$	(9)
11:	$A \vee (B \vee C)$	(10)

The other direction is very similar

Commutatitvy

$\wedge$-commutativity: $T[A \wedge B] \vdash B \wedge A$

1:	$A \wedge B$	premise
2:	A	(1)
3:	B	(1)
4:	$B \wedge A$	(2,3)

The other direction is very similar

Commutatitvy

$\leftrightarrow$-commutativity: $T[A \leftrightarrow B] \vdash B \leftrightarrow A$

1:	$A \leftrightarrow B$	premise
2:	$A \rightarrow B$	(1)
3:	$B \rightarrow A$	(1)
4:	$B \leftrightarrow A$	(2,3)

The other direction is very similar

Commutatitvy

$\vee$-commutativity: $T[A \vee B] \vdash B \vee A$

1:	$\neg(B \vee A)$	assumption
2:	$\neg B \wedge \neg A$	DM1(1)
3:	$\neg B$	(2)
4:	$\neg A$	(2)
5:	$\neg A \wedge \neg B$	(3,4)
6:	$\neg(A \vee B)$	DM2(5)
7:	$A \vee B$	premise
8:	$(A \vee B) \wedge \neg(A \vee B)$	(6,7)
9:	$\neg(B \vee A) \rightarrow (A \vee B) \wedge \neg(A \vee B)$	(1-8)
10:	$\neg\neg(B \vee A)$	(9)
11:	$B \vee A$	(10)

The other direction is very similar

Idempotence

$\vee$-idempotence: $T[A \vee A] \vdash A$

1:	$\neg A$	assumption
2:	$A \vee A$	premise
3:	A	cut1(1,2)
4:	$A \wedge \neg A$	(1,3)
5:	$\neg A \rightarrow A \wedge \neg A$	(1-4)
6:	$\neg\neg A$	(5)
7:	A	(6)

$\vee$-idempotence: $T[A] \vdash A \vee A$

1:	A	premise
2:	$A \vee A$	(1)

Idempotence

$\wedge$-idempotence: $T[A \wedge A] \vdash A$

1:	$A \wedge A$	premise
2:	A	(1)

$\wedge$-idempotence: $T[A] \vdash A \wedge A$

1:	A	premise
2:	$A \wedge A$	(1,1)

Distributivity

$\vee$-distributivity: $T[A \vee (B \wedge C)] \vdash (A \vee B) \wedge (A \vee C)$

1:	$\quad \neg(A \vee B)$	assumption
2:	$\quad \neg A \wedge \neg B$	DM1(1)
3:	$\quad \neg A$	(2)
4:	$\quad \neg B$	(2)
5:	$\quad A \vee (B \wedge C)$	premise
6:	$\quad B \wedge C$	cut(3,5)
7:	$\quad B$	(6)
8:	$\quad B \wedge \neg B$	(4,7)
9:	$\neg(A \vee B) \rightarrow B \wedge \neg B$	(1-8)
10:	$\neg\neg(A \vee B)$	(9)
11:	$A \vee B$	(10)
12:	$\quad \neg(A \vee C)$	assumption
13:	$\quad \neg A \wedge \neg C$	DM1(12)
14:	$\quad \neg A$	(13)
15:	$\quad \neg C$	(13)
16:	$\quad A \vee (B \wedge C)$	premise
17:	$\quad B \wedge C$	cut(14,16)
18:	$\quad C$	(17)
19:	$\quad C \wedge \neg C$	(15,18)
20:	$\neg(A \vee C) \rightarrow C \wedge \neg C$	(12-19)
21:	$\neg\neg(A \vee C)$	(20)
22:	$A \vee C$	(21)
23:	$(A \vee B) \wedge (A \vee C)$	(11,22)

Distributivity

$\vee$-distributivity: $T[(A \vee B) \wedge (A \vee C)] \vdash A \vee (B \wedge C)$

1:	$\neg(A \vee (B \wedge C))$	assumption
2:	$\neg A \wedge \neg(B \wedge C)$	DM1(1)
3:	$\neg A$	(2)
4:	$\neg(B \wedge C)$	(2)
5:	$(A \vee B) \wedge (A \vee C)$	premise
6:	$A \vee B$	(5)
7:	$A \vee C$	(5)
8:	B	cut(3,6)
9:	C	cut(3,7)
10:	$B \wedge C$	(8,9)
11:	$(B \wedge C) \wedge \neg(B \wedge C)$	(4,10)
12:	$\neg(A \vee (B \wedge C)) \rightarrow (B \wedge C) \wedge \neg(B \wedge C)$	(1-11)
13:	$\neg\neg(A \vee (B \wedge C))$	(12)
14:	$A \vee (B \wedge C)$	(13)

Distributivity

$\wedge$-distributivity: $T[A \wedge (B \vee C)] \vdash (A \wedge B) \vee (A \wedge C)$

1:	$A \wedge (B \vee C)$	premise
2:	A	(1)
3:	$B \vee C$	(1)
4:	$\quad \neg((A \wedge B) \vee (A \wedge C))$	assumption
5:	$\quad \neg(A \wedge B) \wedge \neg(A \wedge C)$	DM1(4)
6:	$\quad \neg(A \wedge B)$	(5)
7:	$\quad \neg(A \wedge C)$	(5)
8:	$\quad \neg A \vee \neg B$	DM3(6)
9:	$\quad \neg A \vee \neg C$	DM3(7)
10:	$\qquad \neg A$	assumption
11:	$\qquad A \wedge \neg A$	(2,10)
12:	$\quad \neg A \rightarrow A \wedge \neg A$	(10-11)
13:	$\quad \neg\neg A$	(12)
14:	$\quad \neg B$	cut(8,13)
15:	$\quad \neg C$	cut(9,13)
16:	$\quad \neg B \wedge \neg C$	(14,15)
17:	$\quad \neg(B \vee C)$	DM2(16)
18:	$\quad (B \vee C) \wedge \neg(B \vee C)$	(3,17)
19:	$\neg((A \wedge B) \vee (A \wedge C)) \rightarrow (B \vee C) \wedge \neg(B \vee C)$	(4-18)
20:	$\neg\neg((A \wedge B) \vee (A \wedge C))$	(19)
21:	$(A \wedge B) \vee (A \wedge C)$	(20)

Distributivity

$\wedge$-distributivity: $T[(A \wedge B) \vee (A \wedge C)] \vdash A \wedge (B \vee C)$

1:	$\quad\neg A$	assumption
2:	$\quad\neg A \vee \neg B$	(1)
3:	$\quad\neg(A \wedge B)$	DM4(2)
4:	$\quad(A \wedge B) \vee (A \wedge C)$	premise
5:	$\quad A \wedge C$	cut(3,4)
6:	$\quad A$	(5)
7:	$\quad A \wedge \neg A$	(1,6)
8:	$\neg A \rightarrow A \wedge \neg A$	(1-7)
9:	$\neg\neg A$	(8)
10:	A	(9)
11:	$\quad\neg(B \vee C)$	assumption
12:	$\quad\neg B \wedge \neg C$	(11)
13:	$\quad\neg B$	(12)
14:	$\quad\neg C$	(12)
15:	$\quad\neg A \vee \neg B$	(13)
16:	$\quad\neg(A \wedge B)$	DM4(15)
17:	$\quad(A \wedge B) \vee (A \wedge C)$	premise
18:	$\quad A \wedge C$	cut(16,17)
19:	$\quad C$	(18)
20:	$\quad C \wedge \neg C$	(14,19)
21:	$\neg(B \vee C) \rightarrow C \wedge \neg C$	(11-20)
22:	$\neg\neg(B \vee C)$	(21)
23:	$B \vee C$	(22)
24:	$A \wedge (B \vee C)$	(10,23)

Implication

implication: $T[A \rightarrow B] \vdash \neg A \vee B$

1:	$A \rightarrow B$	premise
2:	$\neg B$	assumption
3:	A	assumption
4:	B	(1,3)
5:	$B \wedge \neg B$	(2,4)
6:	$A \rightarrow B \wedge \neg B$	(3-5)
7:	$\neg A$	(6)
8:	$\neg A \vee B$	(7)
9:	$\neg B \rightarrow \neg A \vee B$	(2-8)
10:	B	assumption
11:	$\neg A \vee B$	(10)
12:	$B \rightarrow \neg A \vee B$	(10-11)
13:	$B \vee \neg B$	tau
14:	$\neg A \vee B$	(9,12,13)

Implication

implication: $T[\neg A \vee B] \vdash A \rightarrow B$

1:	$\neg A \vee B$	premise
2:	A	assumption
3:	$\neg A$	assumption
4:	$A \wedge \neg A$	(2,3)
5:	$\neg A \rightarrow A \wedge \neg A$	(3,4)
6:	$\neg\neg A$	(5)
7:	B	cut(1,6)
8:	$A \rightarrow B$	(2-7)

Implication

double implication: $T[A \leftrightarrow B] \vdash (A \rightarrow B) \wedge (B \rightarrow A)$

1:	$A \leftrightarrow B$	premise
2:	$A \rightarrow B$	(1)
3:	$B \rightarrow A$	(1)
4:	$(A \rightarrow B) \wedge (B \rightarrow A)$	(2,3)

double implication: $T[(A \rightarrow B) \wedge (B \rightarrow A)] \vdash A \leftrightarrow B$

1:	$(A \rightarrow B) \wedge (B \rightarrow A)$	premise
2:	$A \rightarrow B$	(1)
3:	$B \rightarrow A$	(1)
4:	$A \leftrightarrow B$	(2,3)

Double negation

double negation: $T[\neg\neg A] \vdash A$

1:	$\neg\neg A$	premise
2:	A	(1)

double negation: $T[A] \vdash \neg\neg A$

1:	$\neg A$	assumption
2:	A	premise
3:	$A \wedge \neg A$	(1,2)
4:	$\neg A \rightarrow A \wedge \neg A$	(1-3)
5:	$\neg\neg A$	(4)

Contraposition

contraposition: $T[A \rightarrow B] \vdash \neg B \rightarrow \neg A$

1:	$\neg B$	assumption
2:	A	assumption
3:	$A \rightarrow B$	premise
4:	B	(2,3)
5:	$B \wedge \neg B$	(1,4)
6:	$A \rightarrow B \wedge \neg B$	(2-5)
7:	$\neg A$	(6)
8:	$\neg B \rightarrow \neg A$	(1-7)

Contraposition

contraposition: $T[\neg B \rightarrow \neg A] \vdash A \rightarrow B$

1:	A	assumption
2:	$\neg B$	assumption
3:	$\neg B \rightarrow \neg A$	premise
4:	$\neg A$	(2,3)
5:	$A \wedge \neg A$	(1,4)
6:	$\neg B \rightarrow A \wedge \neg A$	(2-5)
7:	$\neg\neg B$	(6)
8:	B	(7)
9:	$A \rightarrow B$	(1-8)