Exercises for Differential calculus in several variables. Bachelor Degree Biomedical Engineering

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Chapter 2.3 Global and Constrained Extrema

Problem 1. Find the extrema of the following functions on the specified domains:

i) $f(x,y) = x^3y^3$ on $\mathbb{R}^2$;

ii) $f(x,y) = x^4y^4$ on $\mathbb{R}^2$;

iii) $f(x,y) = \dfrac{x-y}{1+x^2+y^2}$ on $\mathbb{R}^2$;

iv) $f(x,y) = |x| + |y|$ on $A = \{(x,y):\ |x| \le 1, |y| \le 1\}$.

Solution: *i)* $(x,0)$ and $(0,y)$ are saddle points (f has no minima and no maxima); *ii)* $(x,0)$ and $(0,y)$ are global minima with value 0 (f has no maxima); *iii)* f has at $(1/\sqrt{2}, -1/\sqrt{2})$ a local maximum with value $f(1/\sqrt{2}, -1/\sqrt{2}) = 1/\sqrt{2}$, and at $(-1/\sqrt{2}, 1/\sqrt{2})$ a local minimum with value $f(-1/\sqrt{2}, 1/\sqrt{2}) = -1/\sqrt{2}$; *iv)* $(0,0)$ is a global minimum and $(\pm 1, \pm 1)$ are global maxima.

Problem 2. Solve the following optimization problems constrained to the sphere $x^2+y^2+z^2 = 1$:

i) Maximize the function $f(x,y,z) = xyz$;

ii) Minimize the function $f(x,y,z) = x + 2y + 4z$.

Solution:

i) Maximum at $\frac{1}{\sqrt{3}}(1,1,1)$, $\frac{1}{\sqrt{3}}(-1,-1,1)$, $\frac{1}{\sqrt{3}}(-1,1,-1)$, $\frac{1}{\sqrt{3}}(1,-1,-1)$.

ii) Minimum at $-\frac{1}{\sqrt{21}}(1,2,4)$.

Problem 3. i) Compute the minimum of the function $f(x,y) = x^2+y^2$ on the set $A = \{xy = 1\}$.

ii) Compute the minimum of the function $f(x,y) = xy$ on the set $A = \{x^2+4y^2 = 4\}$.

Solution:

i) Minima at the points $(1,1)$ and $(-1,-1)$;

ii) Maxima at the points $(\sqrt{2}, 1/\sqrt{2})$ and $(-\sqrt{2}, -1/\sqrt{2})$; minima at the points $(\sqrt{2}, -1/\sqrt{2})$ and $(-\sqrt{2}, 1/\sqrt{2})$.

Problem 4. Compute the extrema of the following functions constrained to the given subsets:

i) $f(x,y) = xy$ constrained to $2x+3y-5 = 0$;

ii) $u(x,y) = \dfrac{\log x}{x} + \dfrac{\log y}{y}$ constrained to $x + y = 1$, $x, y > 0$.

iii) $h(x,y,z) = x^2y^4z^6$ constrained to $x + y + z = 1,\ x, y, z > 0$;

Solution:

i) maximum at $(5/4, 5/6)$, there is no minimum;

ii) maximum at $(1/2, 1/2)$, there is no minimum;

iii) maximum at $(1/6, 1/3, 1/2)$, there are no minima.

Problem 5. Compute the absolute maxima and minima of function

$$f(x,y) = x^2 + y^2 + 6x - 8y + 25$$

on the set $D = \{x^2 + y^2 \leq 16\}$.

Solution: Minimum at $(-12/5, 16/5)$; maximum at $(12/5, -16/5)$.

Problem 6. Compute the extrema of the following functions on the given subsets:

i) $f(x,y,z) = x + y + z$ on $S = \{\, 2x^2 + 3y^2 + 6z^2 = 1 \,\}$;

ii) $f(x,y) = x^2 + y^2 - 2x - 2y + 2$ on $T = \{\, y/2 \leq x \leq 3 - \sqrt{2y},\ 0 \leq y \leq 2 \,\}$;

iii) $f(x,y,z) = x^2 + y^2 + z^2$ on $U = \{\, (x,y,z) \in \mathbb{R}^3 \,/\, z \geq x^2 + y^2 - 2 \,\}$.

Solution:

i) $(1/2, 1/3, 1/6)$ is a maximum and $(-1/2, -1/3, -1/6)$ is a minimum;

ii) $(3, 0)$ is a maximum and $(1, 1)$ is a minimum;

iii) The set of points

$$\begin{cases} x^2 + y^2 = \frac{3}{2}, \\ z = -\frac{1}{2}, \end{cases}$$

whose value is $f(x_0, y_0, z_0) = \frac{7}{4}$ will be a set of maximum points and at $(0, 0, 0)$ there will be a minimum.

Problem 7. Find the maximal and minimal values of the function $f(x,y,z) = x + 2y + 3z$ taking into account the two restrictions $x^2 + y^2 = 2,\ x + z = 1$.

Solution: Maximal value $M = 7$, minimal value $m = -1$.

Problem 8. What is the distance of the point $(2, 2, 2)$ to the sphere $x^2 + y^2 + z^2 = 1$.

Solution: $2\sqrt{3} - 1$.

Problem 9. Compute the distance of the point $(4, 4, 10)$ to the sphere $(x-1)^2 + y^2 + (z+2)^2 = 25$ in two ways:

i) use geometrical arguments;

ii) use Lagrange multipliers.

Solution: $d = 8$ with d as the distance.

Problem 10. Express a positive number A as a product of four positive factors, ie. $A = abcd$, with minimal sum.

Solution: $a = A^{1/4}$.

Problem 11. A company produces three different products in quantities Q_1, Q_2, Q_3 and generates a profit given by the expression

$$P(Q_1, Q_2, Q_3) = 2Q_1 + 8Q_2 + 24Q_3.$$

Find the values of Q_1, Q_2, Q_3 that maximize the profit if the production is constrained to

$$Q_1^2 + 2Q_2^2 + 4Q_3^2 = 4.5 \times 10^9.$$

Solution: $Q_1 = 10^4$, $Q_2 = 2Q_1$, $Q_3 = 3Q_1$.

Problem 12. The production of a company is described in terms of the function

$$Q = f(K, L) = K^\alpha L^{1-\alpha},$$

where $0 < \alpha < 1$, K is the amount of capital and L is the amount of man-power used. The unit price of the capital is p and the price of the man-power is q. Compute the proportion between the capital and man-power needed to maximize the production using a budget B.

Solution: $\dfrac{K}{L} = \dfrac{\alpha q}{(1-\alpha)p}$.