Exercises for Differential calculus in several variables. Bachelor Degree Biomedical Engineering

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Chapter 3.1 Multiple Integration

Problem 1. Compute $\iint_Q f$ in the following cases:

i) $f(x,y) = xy(x+y)$, $Q = [0,1] \times [0,1]$,

ii) $f(x,y) = x^3 + 3x^2y + y^3$, $Q = [0,1] \times [0,1]$,

iii) $f(x,y) = \sin^2 x \sin^2 y$, $Q = [0,\pi] \times [0,\pi]$,

iv) $f(x,y) = \sin(x+y)$, $Q = [0,\pi/2] \times [0,\pi/2]$,

v) $f(x,y) = x\sin y - ye^x$, $Q = [-1,1] \times [0,\pi/2]$.

Solution: $i)$ $1/3$; $ii)$ 1; $iii)$ $\pi^2/4$; $iv)$ 2; $v)$ $-\pi^2(e-1/e)/8$.

Problem 2. Sketch the integration region Q and compute $\iint_Q f$ in the following cases:

i) $f(x,y) = x^2 - y$, $Q = \{(x,y) \in \mathbb{R}^2\,,\; x \in [-1,1],\; -x^2 \le y \le x^2\}$,

ii) $f(x,y) = xy - x^3$, $Q = \{(x,y) \in \mathbb{R}^2\,,\; x \in [0,1],\; -1 \le y \le x\}$,

iii) $f(x,y) = 2x - \sin(x^2 y)$, $Q = \{(x,y) \in \mathbb{R}^2\,,\; x \in [-2,2],\; |y| \le |x|\}$,

iv) $f(x,y) = y\;\sin x$, $Q = \{(x,y) \in \mathbb{R}^2\,,\; |x| + |y| \le 1\}$.

Solution: $i)$ $4/5$; $ii)$ $-23/40$; $iii)$ 0; $iv)$ 0.

Problem 3. i) Prove the following inequalities without solving explicitly the integral:

$$4\pi \le \int_D (x^2 + y^2 + 1)\, dx\, dy \le 20\pi\,.$$

Here D is the disc with radius 2 centered at the origin.

ii) The set A is the square $[0,2] \times [1,3]$ and consider the function $f(x,y) = x^2 y$. Prove the following inequalities without solving explicitly the integral:

$$0 \le \int_A f(x,y)\, dx\, dy \le 48\,.$$

iii) Improve the preceding estimations and show

$$3 \le \int_A f(x,y)\, dx\, dy \le 25\,.$$

Hint: divide the set A into four equal squares.

Problem 4. Compute $\int_0^1 \int_0^1 f(x,y)\,dx\,dy$, where $f(x,y) = \max(|x|,|y|)$.

Solution: $2/3$.

Problem 5. Describe the integration region and change the order of integration in the following integrals:

$$i) \quad \int_0^3 \int_{4x/3}^{\sqrt{25-x^2}} f(x,y)dy\,dx \qquad ii) \quad \int_0^1 \int_0^y f(x,y)dx\,dy$$

$$iii) \quad \int_0^{\pi/2} \int_{-\sin(x/2)}^{\sin(x/2)} f(x,y)dy\,dx \qquad iv) \quad \int_1^e \int_0^{\log x} f(x,y)dy\,dx.$$

Solution:

i) $\{0 \leq y \leq 4,\ 0 \leq x \leq 3y/4\} \cup \{4 \leq y \leq 5,\ 0 \leq x \leq \sqrt{25-y^2}\}$;

ii) $\{0 \leq x \leq 1,\ x \leq y \leq 1\}$;

iii) $\{-1/\sqrt{2} \leq y \leq 0,\ -2\arcsin y \leq x \leq \pi/2\} \cup \{0 \leq y \leq 1/\sqrt{2}\,,\ 2\arcsin y \leq x \leq \pi/2\}$;

iv) $\{0 \leq y \leq 1,\ e^y \leq x \leq e\}$.

Problem 6. Consider the functions

$$f(x,y) = \frac{1}{\sqrt{1-x^2}} \quad \text{and} \quad g(x,y) = \sin(y-1),$$

on $R = \{(x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 \leq 1,\ x \geq 0\}$. Use Fubini's theorem to write each integral in two possible ways. Finally, compute the integrals in the most convenient way.

Solution: $\int_R f = 2,\ \int_R g = 0$.

Problem 7. Compute the integral $\int_0^\pi \int_x^\pi \frac{\sin y}{y}\,dy\,dx$.

Solution: 2.

Problem 8. Compute

i) $\displaystyle\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2)\,dxdydz,$

ii) $\displaystyle\int_0^1 \int_0^1 \int_0^1 (x + y + z)^2\,dxdydz.$

Solution: $i)$ 1; $ii)$ $\frac{5}{2}$.

Problem 9. Compute the following integrals:

i) $\int_W x^3\, dV$, with $W = [0,1] \times [0,1] \times [0,1]$.

ii) $\int_W e^{-xy} y\, dV$, with $W = [0,1] \times [0,1] \times [0,1]$.

iii) $\int_W (2x + 3y + z)\, dV$, with $W = [1,2] \times [-1,1] \times [0,1]$.

iv) $\int_W z e^{x+y}\, dV$, with $W = [0,1] \times [0,1] \times [0,1]$.

Solution: $i)\ 1/4;\ ii)\ 1/e;\ iii)\ 7;\ iv)\ (e-1)^2/2$.

Problem 10. Compute the following integral and sketch the integration region:

$$\int_W x^2 \cos x\, dV,$$

where W is the region defined by the planes $z = 0$, $z = \pi$, $y = 0$, $y = 1$, $x = 0$ and $x + y = 1$.

Solution: $\pi(4\sin 1 + 5\cos 1 - 6)$.