Chapter 4.2 Conservative Fields

Problem 1. Consider the force field \( \mathbf{F}(x,y,z) = (\sin y + z, x \cos y + \exp z, x + y \exp z) \).

i) Show that the path integral along any closed piecewise regular curve equals 0.

ii) Compute a potential function for \( \mathbf{F} \), i.e. find a function \( \phi \) such that \( \mathbf{F} = \nabla \phi \).

Solution: i) \( \text{rot } \mathbf{F} = 0 \) and \( \mathbf{F} \) is of class \( C^1 \) on \( \mathbb{R}^3 \). ii) \( \phi(x,y,z) = x(\sin y + z) + ye^z \).

Problem 2. Compute \( \int_{\gamma} \mathbf{F} \), where \( \mathbf{F}(x,y,z) = (2xe^x^2 + y^2, 2yze^x^2 + y^2, e^x^2 + y^2) \) and \( \gamma \) is the curve in \( \mathbb{R}^3 \) defined by \( \mathbf{r}(t) = (t, t^2, t^3), 0 \leq t \leq 1 \).

Solution: \( \exp^2 \).

Problem 3. Consider the following curve in \( \mathbb{R}^3 \), \( \gamma(t) = (e^t + t(1 - e) - 1, \sin(\pi t), \cos(t^2 - t)) \), \( t \in [0, 1] \), and the vector field \( \mathbf{F}(x,y,z) = (y + z + x \sin x^5, x + z + \arctan y, x + y + \sin^2 z) \).

i) Compute \( \int_{\gamma} \mathbf{F} \).

ii) Does a function \( f \) exist such that \( \nabla f = \mathbf{F} \)? Find \( f \), if possible.

Solution: i) \( \int_{\gamma} \mathbf{F} \cdot \mathbf{dr} = 0 \) ii) \( f(x,y,z) = xy + xz + yz - \frac{1}{5} \cos x^5 + y \arctan y - \log \sqrt{1 + y^2} + \frac{1}{2} z - \frac{1}{4} \sin 2z + K \).

Problem 4. Consider the following curve in \( \mathbb{R}^3 \), \( \Gamma = \{ x^2 + y^2 = 1, \ z = y^2 - x^2 \} \), and the vector field \( \mathbf{F}(x,y,z) = (y^3, e^y, z) \).

i) Compute \( \int_{\Gamma} \mathbf{F} \).

ii) Does a function \( f \) exist such that \( \nabla f = \mathbf{F} \)?

Solution: i) \(-3\pi/4\); ii) No.

Problem 5. Find \( a \) and \( b \) such that the vector field

\[
\mathbf{w}(x,y) = e^{2x + 3y} \left( (a \sin x + a \cos y + \cos x), (b \sin x + b \cos y - \sin y) \right)
\]

is conservative and compute a potential function in this case.

Solution: \( a = 2; \ b = 3; \ \varphi(x,y) = e^{2x + 3y}(\sin x + \cos y) + C \).
**Problem 6.** Consider the vector field

\[ \mathbf{F}(x, y) = \left( \frac{\log(xy)}{x}, \frac{\log(xy)}{y} \right), \]

defined for \( x > 0, y > 0 \), and let \( a > 0, b > 0 \) be two constants.

i) Compute \( \int_{\gamma} \mathbf{F} \) where \( \gamma \) is the hyperbola \( xy = a \) for \( x_1 \leq x \leq x_2 \).

ii) If \( A \) is any point on the hyperbola \( xy = a \), \( B \) is any point on the hyperbola \( xy = b \), and \( \gamma \) is any curve of class \( C^1 \) contained in the first quadrant that joins \( A \) with \( B \), show that

\[ \int_{\gamma} \mathbf{F} = \frac{1}{2} \log(ab) \log(b/a). \]

**Solution:** i) \( \mathbf{F} = \nabla f \), with \( f(x, y) = \frac{1}{2} (\log(xy))^2 + c \), \( \int_{\gamma} \mathbf{F} = 0 \).