Excercises Chapter III

Mathematical Methods of Bioengineering

October 18, 2018

Contents

This is a guide to the exercises that you can solve. If you fall short, there are more similar exercises in the books of the subject. You can ask me questions at the end of class or in tutoring.

The idea is that you solve the exercises and do them on the blackboard in class. Each time you go to the board, it will count to the 5% of the final mark. You must participate at least 3 times in order to get the full 5% and at least 6 times to raise the final grade by +0.5 points.

1 Vectors

2 Differentiation in Several Variables

3 Vector Valued Functions

3.1 Parametrized Curves

1. Sketch the image of the following paths indicating the direction when the parameter increases.

   (a) $\vec{\mathbf{c}}(t) = (2t - 1, 3 - t)$, from $-1 \leq t \leq 1$.

   (b) $\vec{\mathbf{c}}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$.

   (c) $\vec{\mathbf{c}}(t) = (t\cos t, t\sin t)$, from $0 \leq t \leq 6\pi$.

   (d) $\vec{\mathbf{c}}(t) = (3\cos t, 2\sin 2t)$, from $0 \leq t \leq 2\pi$. (Give values or use a computer)

   (e) $\vec{\mathbf{c}}(t) = (t, 3t^2 + 1, 0)$.

   (f) $\vec{\mathbf{c}}(t) = (t, t^2, t^3)$.

2. Calculate speed and acceleration of the paths:

   (a) $\vec{\mathbf{x}}(t) = (3t-5, 2t+7)$.
   (b) $\vec{\mathbf{x}}(t) = (5\cos t, 3\sin t)$.
   (c) $\vec{\mathbf{x}}(t) = (t\sin t, t\cos t, t^2)$.
   (d) $\vec{\mathbf{x}}(t) = (e^t, e^{2t}, 2e^t)$.

3. Find the equation for the tangent line to the given path at the indicated value for the parameter:

   (a) $\vec{\mathbf{x}}(t) = (te^{-t}, e^{3t})$ at $t = 0$.
   (b) $\vec{\mathbf{x}}(t) = (4\cos t, -3\sin t, 5t)$ at $t = \pi/3$.
   (c) $\vec{\mathbf{x}}(t) = (\cos e^t, 3-t^2, t)$ at $t = 1$.

4. A malfunctioning rocket is travelling in the space according to the path $\vec{\mathbf{x}}(t) = (e^{2t}, 3t^3 - 2t, t - \frac{1}{t})$ in the hope of reaching a repair station at the point $(7e^4, 35, 5)$. (Here t represents time in minutes and spatial coordinates are measured in miles). At $t = 2$, the rocket's engines suddenly cease. Will the rocket coast into the repair station ? In case not, calculate the distance to the station.

5. Although the path $\mathbf{x} : [0, 2\pi] \to \mathbb{R}^2,\ \mathbf{x}(t) = (\cos t, \sin t)$ may be the most familiar way to give a parametric description of a unit circle, in this problem you will develop a different set of parametric equations that gives the x- and y-coordinates of a point on the circle in terms of rational functions of the parameter. (This particular parametrization turns out to be useful in the branch of mathematics known as *number theory*.)

   To set things up, begin with the unit circle $x^2 + y^2 = 1$ and consider all lines through the point $(0, -1)$. Note that every line other than the horizontal line $y = -1$ intersects the circle at a point (x, y) other than $(0, -1)$. Let the parameter t be the slope of the line joining $(0, -1)$ and a point (x, y) on the circle.

   (a) Give an equation for the line of slope t joining (0,-1) and (x, y). (Your answer should involve x, y, and t).
   (b) Use your answer in part (a) to write y in terms of x and t. Then substitute this expression for y into the equation for the unit circle. Solve the resulting equations for x in terms of t. Your answer(s) for x will give the points of intersection of the line and the circle.
   (c) Use your result in part (b) to give a set of parametric equations for points (x, y) on the unit circle.
   (d) Does your parametrization in part (c) cover the entire circle? Which, if any, points are missed ?

3.2 Arclength and Differential Geometry

1. Calculate the length of the paths:

   (a) $\mathbf{x}(t) = (2t+1, 7-3t)$ for $-1 \le t \le 2$.
   (b) $\mathbf{x}(t) = (\cos 3t, \sin 3t, 2t^{3/2})$ for $0 \le t \le 2$.
   (c) $\mathbf{x}(t) = (\ln\cos t, \cos t, \sin t)$ for $\pi/6 \le t \le \pi/3$.

Note: $\int \sec x \; dx = \ln |\sec x + \tan x| + C$

2. Consider the path $\mathbf{x} = |t-1|\mathbf{i} + |t|\mathbf{j}$, $-2 \le t \le 2$.
   (a) Sketch this path.
   (b) The path fail to be of class C^1 but is piecewise C^1. Explain.
   (c) Calculate the length of the path.
3. Consider the curve $\mathbf{x}(t) = e^{at}\cos bt\mathbf{i} + e^{at}\sin bt\mathbf{j} + e^{at}\mathbf{k}$.
   (a) Find the arclength parameter $s = s(t)$ for the path.
   (b) Express the original parameter t in terms of s and, thereby, reparametrize x in terms of s.
4. Consider the formula for the curvature $\kappa = \dfrac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}$, where $\mathbf{v}$ is the velocity vector and $\mathbf{a}$ the acceleration vector.
   (a) Use the formula to establish the following well-known result for the curvature of a plane curve $y = f(x)$:
   $$\kappa = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$
   (b) Use your result in a) to find the curvature of $y = \ln(\sin x)$.

3.3 Vector Fields: An Introduction

1. Sketch the given vector field on $\mathbb{R}^2$.
   (a) $\mathbf{F} = (y, -x)$
   (b) $\mathbf{F} = (x, -y)$
   (c) $\mathbf{F} = (x, x^2)$
   (d) $\mathbf{F} = (y^2, y)$
2. Sketch the given vector field on $\mathbb{R}^3$.
   (a) $\mathbf{F} = (3, 2, 1)$
   (b) $\mathbf{F} = (y, -x, 0)$
   (c) $\mathbf{F} = (0, z, -y)$
   (d) $\mathbf{F} = \dfrac{y}{\sqrt{x^2+y^2+z^2}}\mathbf{i} - \dfrac{x}{\sqrt{x^2+y^2+z^2}}\mathbf{j} + \dfrac{z}{\sqrt{x^2+y^2+z^2}}\mathbf{k}$
3. Verify that the path given is a flow line of the indicated vector field.
   (a) $\mathbf{x}(t) = (\sin t, \cos t, 0)$, $\mathbf{F} = (y, -x, 0)$.
   (b) $\mathbf{x}(t) = (\sin t, \cos t, 2t)$, $\mathbf{F} = (y, -x, 2)$.
   (c) $\mathbf{x}(t) = (\sin t, \cos t, e^{2t})$, $\mathbf{F} = (y, -x, 2z)$.
4. Calculate the flow line $\mathbf{x}(t)$ of the given vector field $\mathbf{F}$ that passes through the indicated point at the line specified value of t.

(a) $\mathbf{F}(x, y) = -x\mathbf{i} + y\mathbf{j}$; $\mathbf{x}(0) = (2, 1)$.

(b) $\mathbf{F}(x, y) = (x^2, y)$; $\mathbf{x}(1) = (1, e)$.

(c) $\mathbf{F}(x, y) = -2\mathbf{i} - 3y\mathbf{j} + z^3\mathbf{k}$; $\mathbf{x}(0) = (3, 5, 7)$.

5. Consider the vector field $\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} - 3\mathbf{k}$.

   (a) Show that $\mathbf{F}$ is a gradient field.

   (b) Describe the equipotential surfaces of $\mathbf{F}$ in words and with sketches.

6. If $\mathbf{x}$ is a flow line of a gradient vector field $\mathbf{F} = \nabla f$, show that the function $G(t) = f(\mathbf{x}(t))$ is an increasing function of t. (Hint: Show that $G'(t)$ is always nonnegative). Thus, we see that a particle travelling along a flow line of the gradient field $\mathbf{F} = \nabla f$ will move from lower to higher values of the potential function f. That's why physicists define a potential function of a gradient vector field F to be a function g such that $\mathbf{F} = -\nabla g$ (i.e., so that particles travelling along flow lines move from higher to lower values of g).

3.4 Gradient, Divergence, Curl and Del

1. Calculate the divergence of the vectors:

   (a) $\mathbf{F} = (x^2, y^2)$.

   (b) $\mathbf{F} = (x + y, y + z, x + z)$.

   (c) $\mathbf{F} = (x_1^2, 2x_1^2, ..., nx_1^2)$.

2. Calculate the curl of the vectors:

   (a) $\mathbf{F} = (x^2, -xe^y, 2xyz)$.

   (b) $\mathbf{F} = (x + yz, y + xz, z + xy)$.

   (c) $\mathbf{F} = (\cos yz - x, \cos xz - y, \cos xy - z)$.

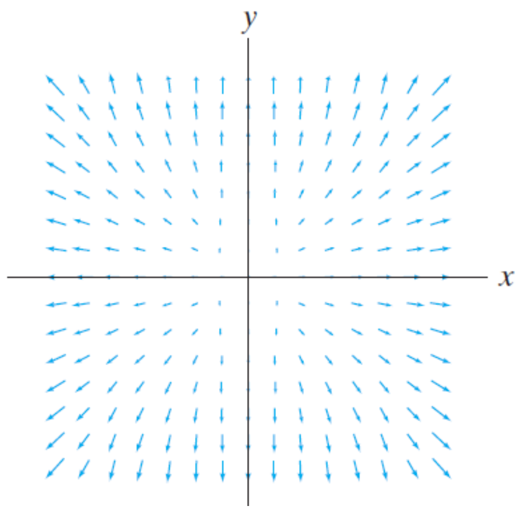
3. Consider the vector field $\mathbf{F_1} = (x, y, z)$ and its curl.

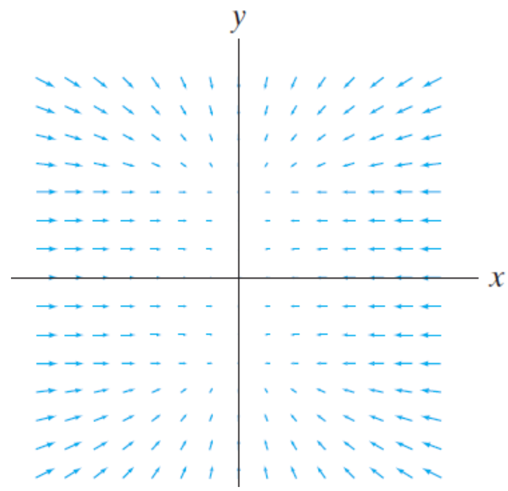
   (a) Sketch the vector field and use your picture to explain geometrically why the curl is as you calculated.

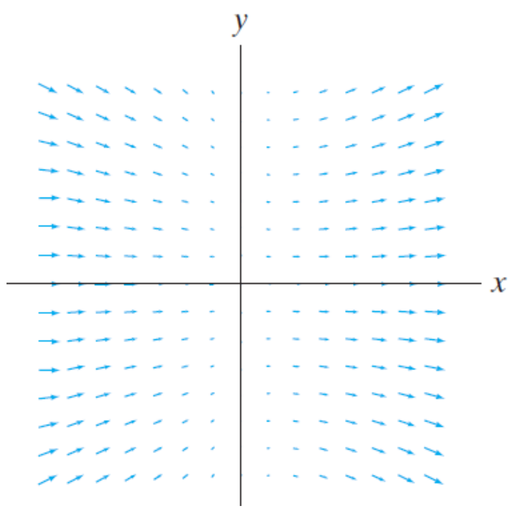
   (b) Use geometry to determine $\nabla \times \mathbf{F_2}$, where $\mathbf{F_2} = \dfrac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}$.

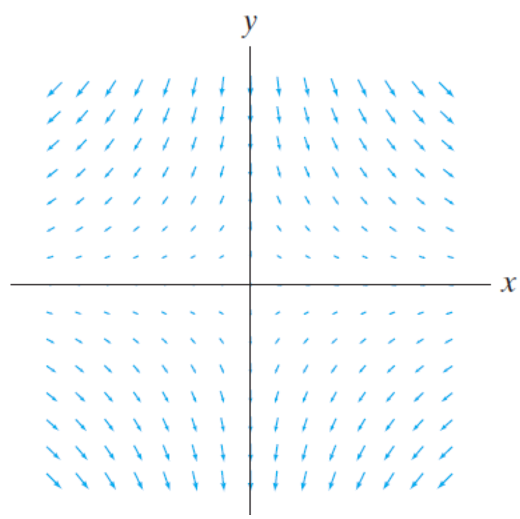
   (c) Verify your answer in part b) by computing explicitly $\nabla \times \mathbf{F_2}$.

4. Can you tell in what portions of $\mathbb{R}^2$, the vector fields shown in the next figures have positive divergence ? Negative divergence?









5. Verify:

   (a) $\nabla r^n = n r^{n-2}\mathbf{r}$.

   (b) $\nabla(\ln r) = \dfrac{\mathbf{r}}{r^2}$.

   (c) $\operatorname{div}(r^n\mathbf{r}) = (n+3)r^n$.