Métodos Matemáticos de Bioingeniería
Grado en Ingeniería Biomédica
Curso 2018-2019

Exercises for solving - Chapter II

1. Find the domain and range of the function

$$\mathbf{f}(s,t) = \left(s+t, \frac{1}{t-1}, s^2+t^2\right)$$

2. Let $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined as:

$$\mathbf{f}(\mathbf{x}) = \mathbf{x} + 3\mathbf{j}$$

Write the components of $\mathbf{f}$ in terms of the components of the vector $\mathbf{x}$.

3. Consider the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$f(\mathbf{x}) = A\mathbf{x}$$

where the matrix A is

$$A = \begin{bmatrix} 2 & -1 \\ 5 & 0 \\ -6 & 3 \end{bmatrix}$$

and the vector $\mathbf{x} \in \mathbb{R}^2$ is written as the following column matrix 2×1

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

   (a) Find an explicit form of the components of $\mathbf{f}$ in terms of the components x_1, x_2 of the vector $\mathbf{x}$.
   (b) Describe the range of $\mathbf{f}$.

4. Evaluate the following limit or explain why does not exist

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2}$$

5. Evaluate the following limit or explain why does not exist

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^4}{x^2 + y^4}$$

6. Evaluate the following limit or explain why does not exist

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2}$$

7. Consider the limit of $f(x,y) = x^4y^4/(x^2+y^4)^3$ as (x,y) approaches $(0,0)$.

   a) (**1 points**) Examine the behaviour of the limit along straight lines of the form $y = mx$. What might the limit be?
   b) (**1 points**) Consider what happens when (x,y) approaches (0,0) along the curve $x = y^2$. Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? Why or why not?

8. Evaluate the following limit or explain why does not exist

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + z^2}}$$

9. Determine whatever the following function is continuous or not in his domain

$$g(x,y) = \begin{cases} \frac{x^3+x^2+xy^2+y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 2 & \text{if } (x,y) = (0,0) \end{cases}$$

10. Demonstrate that the function $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\mathbf{f}(\mathbf{x}) = (6\mathbf{i} - 5\mathbf{k}) \times \mathbf{x}$$

is continuous.

11. Evaluate the first order partial derivative of the following function

$$F(x,y,z) = e^{ax} \cos by + e^{az} \sin bx$$

12. Find the gradient $\nabla f(\mathbf{a})$, where f and $\mathbf{a}$ are given by

$$f(x,y,z) = \cos z \ln(x+y^2), \quad \mathbf{a} = (e, 0, \pi/4)$$

13. Find the matrix of partial derivatives or Jacobian matrix $D\mathbf{f}(\mathbf{a})$ where $\mathbf{f}$ and $\mathbf{a}$ are given by

$$\mathbf{f}(x,y) = (x^2 y, x + y^2, \cos \pi xy), \quad \mathbf{a} = (2, -1)$$

14. Explain why this function is differentiable at all his domain

$$\mathbf{f}(x,y) = \left(\frac{xy^2}{x^2+y^4}, \frac{x}{y} + \frac{y}{x} \right)$$

15. Find the equation of the tangent plane of

$$z = x^2 - 6x + y^3$$

that are parallel to the plain

$$\pi : 4x - 12y + z = 7$$

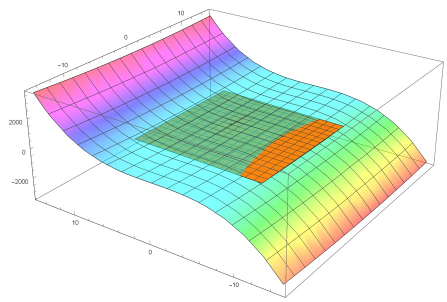


Figura 1: One tangent plane.

16. Find second order partial derivatives (including mixes ones) of the function

$$f(x,y,z) = e^{ax} \sin y + e^{bx} \cos z$$

17. Consider

$$f(x, y, z) = x^7 y^2 z^3 - 2x^4 yz$$

(a) What is $\frac{\partial^4 f}{\partial x^2 \partial y \partial z}$?

(b) What is $\frac{\partial^5 f}{\partial x^3 \partial y \partial z}$?

(c) What is $\frac{\partial^{15} f}{\partial x^{13} \partial y \partial z}$?

18. Consider the partial differential equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

It is know as the **Laplace equation**. Any function f of class C^2 that satisfies the equation is said to be an **harmonic function**.

(a) Is $f(x, y, z) = x^2 + y^2 - 2z^2$ harmonic?

(b) Is $f(x, y, z) = x^2 - y^2 + z^2$ harmonic?

19. Suppose that a bird flies along the helical curve

$$\begin{cases} x = 2\cos t \\ y = 2\sin t \\ z = 3t \end{cases}, t \in \mathbb{R}$$

The bird suddenly encounters a weather front, so that the barometric pressure is varying rather wildly from point to point as

$$P(x, y, z) = \frac{6x^2 z}{y} \text{ atm}$$

Determine how the pressure is changing at

$$t = \frac{\pi}{4} \text{ min}$$

20. The Centers for Disease Control and Prevention provides information on the **Body Mass Index** (BMI). The goal is to give a more meaningful assessment of a person's weight. The BMI is given by the formula

$$\text{BMI} = \frac{10,000w}{h^2}$$

where

- w is an individual's mass in kilograms, and
- h is the person's height in centimeters

While monitoring a child's growth, you estimate that at the time he turned 10 years old, his height showed a growth rate of 0.6 cm and his mass showed a growth rate of 0.4 kg per month Suppose that he was 140 cm tall and weighed 33 kg on his tenth birthday. At what rate is his BMI changing on his tenth birthday?

21. This exercise shows that the chain rule can not be applied if f is not differentiable. Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

Show that:

a) The partial derivatives exists at $(0,0)$.

b) If $\mathbf{g}(t) = (at, bt)$ for some constants a, b, then $f \circ \mathbf{g}$ is derivable but the chain rule is false: $(f \circ \mathbf{g})'(0) \neq \nabla f(0,0) \cdot \mathbf{g}'(0)$.

22. Let

$$\begin{aligned} f(x,y) &= ye^x \\ \mathbf{g}(s,t) &= (s-t, s+t) \end{aligned}$$

Calculate $D(f \circ \mathbf{g})$

23. Let

$$\begin{aligned} \mathbf{f}(x,y) &= \left(xy - \frac{y}{x}, \frac{x}{y} + y^3\right) \\ \mathbf{g}(s,t) &= \left(\frac{s}{t}, s^2 t\right) \end{aligned}$$

Calculate $D(\mathbf{f} \circ \mathbf{g})$

24. Let

$$f(x,y,z) = xyz, \quad \mathbf{a} = (-1,0,2), \quad \mathbf{u} = \frac{2\mathbf{k} - \mathbf{i}}{\sqrt{5}}$$

Calculate the directional derivative of the given function f at the point $\mathbf{a}$ in the direction parallel to the vector $\mathbf{u}$

25. The surface of Lake Erehwon can be represented by a region D in the xy-plane. The lake's depth (in meters) at the point (x, y) is given by the expression

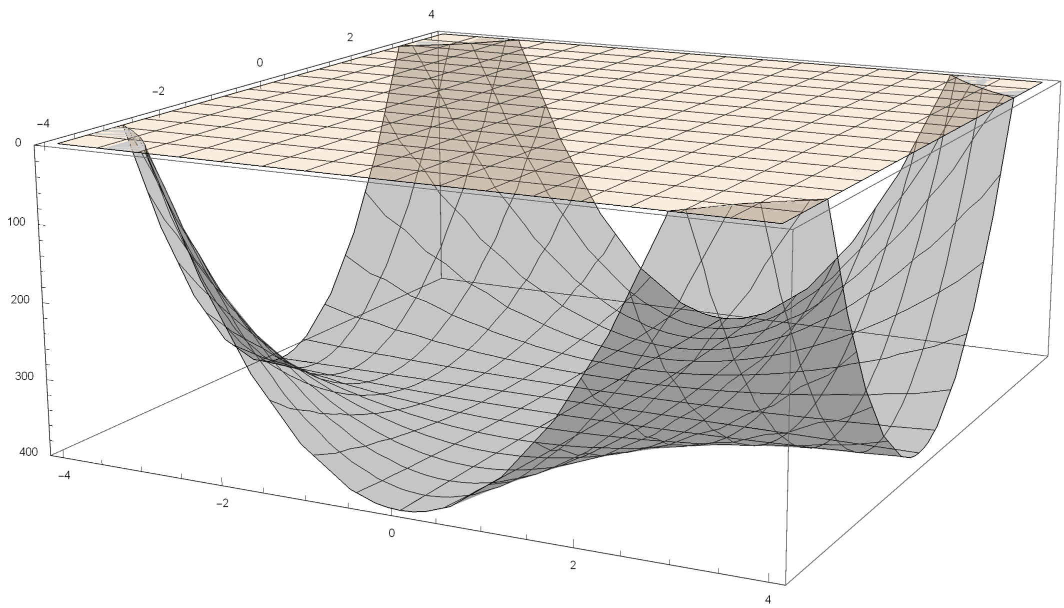
$$h(x, y) = 400 - 3x^2y^2$$

Assume your calculus instructor is in the water at the point $(1, -2)$.

(a) In which direction should she swim so that the depth increases most rapidly (i.e., so that she is most likely to drown)?

(b) In which direction should she swim so that the depth remains constant?

(c) In which direction should she swim so the change of depth increases with a slope of 50 %



Note: A slope of 50 % means that every 100 steps is advanced horizontally is climbed 50 vertically.