Ejemplo. Considere la aplicación lineal $T: \mathbb{R}^5 \to \mathbb{R}^3$ definida por $T(\bar{x}) = A\bar{x}$ donde $A = \begin{pmatrix} 1 & 4 & 7 & 5 & 11 \\ 2 & 5 & 8 & 7 & 13 \\ 3 & 6 & 10 & 9 & 16 \end{pmatrix}$.

Claramente, la matriz asociada con T respecto de las bases estándar ε_5 y ε_3 es $M_T^{\varepsilon_3, \varepsilon_5} = A$.

Considere las nuevas bases de $\mathbb{R}^5$ y $\mathbb{R}^3$

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 10 \end{pmatrix} \right\}$$

$$\mathbb{R}^3 \longleftarrow \mathbb{R}^5$$

$$\begin{tikzcd}
{[T\bar{x}]_{\mathcal{E}_3}} \arrow[d, "\underset{C \leftarrow \mathcal{E}_3}{P}"'] & = M_T^{\mathcal{E}_3, \mathcal{E}_5} & {[\bar{x}]_{\mathcal{E}_5}} \\
{[T\bar{x}]_C} & = M_T^{C,B} & {[\bar{x}]_B} \arrow[u, "\underset{\mathcal{E}_5 \leftarrow B}{P}"']
\end{tikzcd}$$

$$M_T^{C,B} = \underset{C \leftarrow \mathcal{E}_3}{P} \; M_T^{\mathcal{E}_3, \mathcal{E}_5} \; \underset{\mathcal{E}_5 \leftarrow B}{P} \qquad\qquad M_T^{\mathcal{E}_3, \mathcal{E}_5} = A$$

$$\underset{\mathcal{E}_5 \leftarrow B}{P} = \left([\bar{b}_1]_{\mathcal{E}_5} \; [\bar{b}_2]_{\mathcal{E}_5} \; [\bar{b}_3]_{\mathcal{E}_5} \; [\bar{b}_4]_{\mathcal{E}} \; [\bar{b}_5]_{\mathcal{E}_5} \right)$$

$$= \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underset{\mathcal{E}_3 \leftarrow C}{P} = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{pmatrix} \qquad \underset{C \leftarrow \mathcal{E}_3}{P} = \underset{\mathcal{E}_3 \leftarrow C}{P^{-1}}$$

$$\left(\begin{array}{cccccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 2 & 5 & 8 & 0 & 1 & 0 \\ 3 & 6 & 10 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -11 & -3 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccccc} 1 & 4 & 0 & -6 & 14 & -7 \\ 0 & -3 & 0 & 4 & -11 & 6 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$\sim \begin{pmatrix} 1 & 4 & 0 & -6 & 14 & -7 \\ 0 & 1 & 0 & -4/3 & 11/3 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -2/3 & -2/3 & 1 \\ 0 & 1 & 0 & -4/3 & 11/3 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{pmatrix}$$

$$\underset{C\leftarrow\mathcal{E}_3}{P} = \begin{pmatrix} -2/3 & -2/3 & 1 \\ -4/3 & 11/3 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

$$\underset{3\times 5}{M_T^{C,B}} = \overset{[T\bar{x}]_C}{\underset{3\times 3}{\begin{pmatrix} -2/3 & -2/3 & 1 \\ -4/3 & 11/3 & -2 \\ 1 & -2 & 1 \end{pmatrix}}} \underbrace{\overset{[T\bar{x}]_{\mathcal{E}_3}}{\underset{3\times 5}{\begin{pmatrix} 1 & 4 & 7 & 5 & 11 \\ 2 & 5 & 8 & 7 & 13 \\ 3 & 6 & 10 & 9 & 16 \end{pmatrix}}}}_{[T\bar{x}]_{\mathcal{E}_3} \leftarrow [\bar{x}]_{\mathcal{E}_5}} \overset{[\bar{x}]_{\mathcal{E}_5} \longrightarrow [\bar{x}]_B}{\underset{5\times 5}{\begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}}$$

Ejemplo $D:\mathbb{P}_2 \to \mathbb{P}_2$ $\qquad Dp(x) = p'(x)$

$\mathcal{E} = \{1, x, x^2\}$

$$M_D^{\mathcal{E},\mathcal{E}} = M_D^{\mathcal{E}} = \left([D1]_{\mathcal{E}} \;\; [Dx]_{\mathcal{E}} \;\; [Dx^2]_{\mathcal{E}} \right)$$
$$= \left([0]_{\mathcal{E}} \;\; [1]_{\mathcal{E}} \;\; [2x]_{\mathcal{E}} \right)$$
$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$p(x) = a + bx + cx^2$$
$$= p(1) + p'(1)(x-1) + \frac{p''(1)}{2!}(x-1)^2$$

$T = \{1, \; x-1, \; (x-1)^2\}$

$M_D^{T,T}$: 1. Directamente de la fórmula.

2. Utilizar $M_D^{\mathcal{E}}$ y cambios de base.

$$M_D^{T} = \left([D1]_T \;\; [D(x-1)]_T \;\; [D(x-1)^2]_T \right)$$
$$= \left([0]_T \;\; [1]_T \;\; [2(x-1)]_T \right)$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

2. $M_D^T = \underset{T \leftarrow \mathcal{E}}{P} \, M_D^{\mathcal{E}} \, \underset{\mathcal{E} \leftarrow T}{P}$

$\mathcal{E}$

$M_D^{T,\mathcal{E}}$: 1. Directamente de la fórmula.

2. Utilizar $M_D^{\mathcal{E}}$ y cambios de base.

$$M_D^{T,\mathcal{E}} = \left([D1]_T \;\; [Dx]_T \;\; [Dx^2]_T \right)$$

$$= \left([0]_T \;\; [1]_T \;\; [2x]_T \right)$$

$$= \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$f(x) = 2x = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$$

$$= 2 + 2(x-1)$$

$$2.\ M^{T,\mathcal{E}} = \underset{T \leftarrow \mathcal{E}}{P}\ M_D^{\mathcal{E},\mathcal{E}}\ \underbrace{\underset{\mathcal{E} \leftarrow \mathcal{E}}{P}}_{}{}^{= I_3} = \underset{T \leftarrow \mathcal{E}}{P}\ M_D^{\mathcal{E},\mathcal{E}}$$

$$\underset{T \leftarrow \mathcal{E}}{P} = \left([1]_T \quad [x]_T \quad [x^2]_T \right)$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$x = 1 + (x-1)$$

$$x^2 = (x-1)^2 + 2(x-1) + 1$$

$$M_D^{T,\mathcal{E}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$p(x) = 1 + x^2 \qquad p'(x) = 2x$

$$[p']_T = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

$p' = 2 + 2(x-1) = 2x$