## C.6 Continuity

**Problem 6.1**

(a) The function $g(x) = |x|$ is a continuous function and $|f(x)| = (g \circ f)(x)$ is continuous because the composition of continuous functions is a continuous function.
As for the reciprocal, take

$$f(x) = \begin{cases} 1, & x \geqslant 0, \\ -1, & x < 0. \end{cases}$$

It is clearly a discontinuous function, however $|f(x)| = 1$ everywhere, which is continuous. This example illustrates that from the fact that $|f(x)|$ is continuous one cannot conclude that $f(x)$ itself is continuous.

(b) We are talking here about a function $f : \mathbb{R} \mapsto \mathbb{Q}$ that is continuous. One such function would necessarily be constant. Let us see why. Suppose that $f(x_1) = q_1$ and $f(x_2) = q_2 \neq q_1$. Since the function is continuous it must take all intermediate values between $q_1$ and $q_2$ within the interval $[x_1, x_2]$. But between any two rational numbers there are infinitely many irrational numbers, so there must exist $x \in (x_1, x_2)$ such that $f(x)$ is irrational. This is a contradiction and therefore $q_2 \neq q_1$ is not possible.

**Problem 6.2**

(a) The information that the function is surjective means that $x_0$ and $x_1$ in $[0, 1]$ such that $f(x_0) = 0$ and $f(x_1) = 1$. Now, consider the interval $[x_0, x_1]$ (or $[x_1, x_0]$, depending on which one is bigger). The function $g(x) = f(x) - x$ is continuous (the sum of two continuous functions) and satisfies

$$g(x_0) = -x_0, \qquad g(x_1) = 1 - x_1.$$

If $x_0 = 0$ then $c = 0$ is the point we are looking for. If $x_1 = 1$ then $c = 1$ is that point. If none of these two things happen then $g(x_0) < 0$ and $g(x_1) > 0$ and we can apply Bolzano's theorem: there must exist $c \in (0, 1)$ such that $g(c) = 0$ —which is equivalent to $f(c) = c$. Whichever the case, we can conclude that there exists $c \in [0, 1]$ such that $f(c) = c$.

(b) Consider the number

$$\mu = \frac{1}{n} \sum_{k=1}^{n} f(x_k).$$

We can obtain a lower bound to $\mu$ by replacing in this expression all the $f(x_k)$ by the smallest one. Thus,

$$\mu \geqslant \min_{k=1,\dots,n} f(x_k).$$

Likewise, we can obtain an upper bound replacing them by the largest one:

$$\mu \leqslant \max_{k=1,\dots,n} f(x_k).$$

So $\mu$ is a value intermediate between two values that the function $f$ takes in the interval $[a, b]$, therefore, since it is continuous, there must be a number $c \in [a, b]$ at which $f(c) = \mu$.

**Problem 6.3** Since $f$ is a rational function, all that it is required for it to be continuous is that the denominator does not vanish within the specified set.

(a) In this case the denominator must never vanish. If $\lambda = 0$ the function $f(x) = 1$ and trivially continuous in $\mathbb{R}$. Consider now $\lambda \neq 0$. Since in this case the denominator is a quadratic polinomial, the requirement that it never vanishes can be rephrase as its two roots being complex. The condition for that is that the discriminant is negative, so

$$4\lambda^2 - 4\lambda < 0 \quad \Leftrightarrow \quad \lambda(\lambda - 1) < 0.$$

This holds if each factor has a different sign, i.e., if $0 < \lambda < 1$. Therefore the function is continuous in $\mathbb{R}$ provided $\lambda \in [0,1)$.

(b) Any of the values of $\lambda$ found in the previous item make the function continuous in $\mathbb{R}$ —hence also in $[0,1]$—, so we just have to check what happens if $\lambda < 0$ or if $\lambda \geqslant 1$. In any of these two cases the denominator will have two real roots, so the key point is that none of them lies within the interval $[0,1]$ where we want $f(x)$ to be continuous.
By solving the quadratic equation we find the two roots as

$$x_1 = \frac{\lambda + \sqrt{\lambda(\lambda-1)}}{\lambda} = 1 + \sqrt{1-\lambda^{-1}}, \qquad x_2 = \frac{\lambda - \sqrt{\lambda(\lambda-1)}}{\lambda} = 1 - \sqrt{1-\lambda^{-1}}.$$

If $\lambda = 1$ both $x_1 = x_2 = 1$ and so $f$ is not continuous at $x = 1$. Thus $\lambda \neq 1$ is required. In this case $x_1 > 1$, so it will always be outside the interval $[0,1]$. We can ignore it. On the contrary, $x_2 < 1$, so it will be also ouside the interval provided $x_2 < 0$. This condition implies $\sqrt{1-\lambda^{-1}} > 1$, which can only hold if $\lambda < 0$.
Summarising, $f(x)$ will be continuous in $[0,1]$ provided $\lambda < 1$.

**Problem 6.4**

(i) Numerator and denominator are continuous functions in $\mathbb{R}$, so this function will be continuous except when the denominator vanishes. It does when $x^2 - 8x + 12 = (x-6)(x-2) = 0$, so $f$ is continuous in $\mathbb{R} - \{2,6\}$.

(ii) The function is the sum of a plynomial (continuous in $\mathbb{R}$) and the function $e^{3/x}$. The exponential is continuous everywhere and the function $3/x$ too, except for $x = 0$. Besides,

$$\lim_{x \to 0^+} e^{3/x} = \infty,$$

so $f$ is continous in $\mathbb{R} - \{0\}$.

(iii) Polynomials are continuous in $\mathbb{R}$ and so the tangent except when its argument is an odd multiple of $\pi/2$. This means the points

$$3x + 2 = n\pi + \frac{\pi}{2} \quad \Rightarrow x = \frac{n\pi - 2}{3} + \frac{\pi}{6}, \quad n \in \mathbb{Z}.$$

$f$ is continuous except at these infinitely many points.

(iv) The polynomial is continuous in $\mathbb{R}$, so $f$ is continuous wherever the argument of the square root is not negative. This means $x^2 - 5x + 6 = (x-3)(x-2) \geqslant 0$, which happens for $x \geqslant 3$ or $x \leqslant 2$. Thus $f$ is continuous in $(-\infty, 2) \cup (3, \infty)$.

(v) $\arcsin x$ is only defined for $x \in [-1,1]$, but in this region it is continuous because is the inverse of a continuous function. Thus $f$ is continuous in $[-1,1]$.

(vi) The polynomials are continuous everywhere, so the only requirement is that the argument of the logarithm is positive, i.e., $8x - 3 > 0$. Hence $f$ is continuous in $(3/8, \infty)$.

(vii) This function represents the decimal part of $x$ and is clearly discontinuous at the integers. Thus $f$ is continuous in $\mathbb{R} - \mathbb{Z}$.

(viii) The polynomial and the sine function are both continuous everywhere, and so is $1/x$ except at $x = 0$. Function $f$ is defined at $x = 0$ though, so we must check the definition of continuity at this specific point. Since $|x^2 \sin(1/x)| \leqslant x^2$ and $x^2 \to 0$ as $x \to 0$, then

$$\lim_{x\to 0} f(x) = 0 = f(0)$$

and $f$ is continuous in $\mathbb{R}$.

(ix) For $x > 0$ the function is continuous except for $x = (2n-1)\pi/2$, $n \in \mathbb{N}$. For $x < 0$ the function is always continuous. We must compute the two one-sided limits at $x = 0$ to check for continuity at that point. Now,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\tan x}{\sqrt{x}} = \lim_{x\to 0^+} \frac{x}{\sqrt{x}} = \lim_{x\to 0^+} \sqrt{x} = 0.$$

And on the other side,

$$\lim_{x\to 0^-} e^{1/x} = \lim_{t\to -\infty} e^t = 0.$$

Thus,

$$\lim_{x\to 0} f(x) = 0 = f(0),$$

so $f$ is continuous in $\mathbb{R} - \{(2n-1)\pi/2 : n \in \mathbb{N}\}$.

(x) As close as we like to a rational number there is always an irrational number. As close as we like to an irrational number there is always a rational number. So, $f$ is discontinuous at every $x \neq 0$. At $x = 0$ function $f(x)$ is continuous though. The reason is that $|f(x)| = |x| \to 0$ as $x \to 0$, so

$$\lim_{x\to 0} f(x) = 0 = f(0).$$

(xi) Each piece of this piecwise function separately is a continuous function, so we just need to check what happens at the joints. Thus,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-1)^3 = 0, \qquad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (|x| - x) = 0,$$

so

$$\lim_{x\to 1} f(x) = 0 = f(1).$$

And

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} (|x| - x) = 2, \qquad \lim_{x\to -1^+} f(x) = \lim_{x\to -1^-} \sin(\pi x) = 0,$$

so $f(x)$ is continuous in $\mathbb{R} - \{-1\}$.

(xii) The two polynomials defining the function for $|x| \geqslant 1$ are continuous function. In $(-1,1)$ the function is defined as $\operatorname{sgn} x + 1$, which is continuous except at $x = 0$. We now need to check the two joints. Thus,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x = 2, \qquad \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (\operatorname{sgn} x + 1) = 2,$$

so

$$\lim_{x\to 1} f(x) = 2 = f(1).$$

And

$$\lim_{x\to-1^-} f(x) = \lim_{x\to-1^+} (\operatorname{sgn} x + 1) = 0, \qquad \lim_{x\to-1^+} f(x) = \lim_{x\to-1^-} (x+1)^2 = 0,$$

so

$$\lim_{x\to-1} f(x) = 0 = f(-1).$$

Summarising, $f(x)$ is continuous in $\mathbb{R} - \{0\}$.

(xiii) Each of the three pieces of this piecewise function is continuous (a polynomial or the absolute value of a polynomial), so we need to check just the joints. Thus,

$$\lim_{x\to2^+} f(x) = \lim_{x\to2^+} (4x-5) = 3, \qquad \lim_{x\to2^-} f(x) = \lim_{x\to2^-} |x^2-1| = 3,$$

so

$$\lim_{x\to2} f(x) = 3 = f(2).$$

And

$$\lim_{x\to-2^-} f(x) = \lim_{x\to-2^+} |x^2-1| = 3, \qquad \lim_{x\to-2^+} f(x) = \lim_{x\to-2^-} x^2 = 4,$$

so $f(x)$ is continuous in $\mathbb{R} - \{-2\}$.

(xiv) The functions defining $f(x)$ for $|x| > 1$ are both polynomials —hence continuous. Within $|x| \leqslant 1$ it is defined as $g(x) = x - \lfloor x \rfloor$. Now, $g(x) = x+1$ for all $-1 \leqslant x < 0$, $g(x) = x$ for all $0 \leqslant x < 1$, and $g(1) = 0$. Thus function $f(x)$ can be redefined as

$$f(x) = \begin{cases} (x-1)^2, & x \geqslant 1, \\ x, & 0 \leqslant x < 1, \\ x+1, & x < 0. \end{cases}$$

All three pieces are continuous (polynomials), so we must look at the joints. So,

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} (x-1)^2 = 0, \qquad \lim_{x\to1^-} f(x) = \lim_{x\to1^-} x = 1,$$

and

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} x = 0, \qquad \lim_{x\to0^-} f(x) = \lim_{x\to0^-} (x+1) = 1.$$

Therefore the $f(x)$ is continuous in $\mathbb{R} - \{0,1\}$.

**Problem 6.5**

(i) Denoting $f(x) = x^2 - 18x + 2$, a continuous function in $\mathbb{R}$, we have $f(-1) = 21$, $f(1) = -15$, so Bolzano's theorem guarantees at least one zero in $[-1,1]$.

(ii) Denoting $f(x) = x - \sin x - 1$, a continuous function in $\mathbb{R}$, we have $f(0) = -1$ and $f(\pi) = \pi - 1 > 0$, so Bolzano's theorem guarantees at least one zero in $[0,\pi]$.

(iii) Since $e^x > 0$, we know that $e^x + 1 > 0$, so the equation cannot have any solution in $\mathbb{R}$.

(iv) Since $-1 \leqslant \cos x \leqslant 1$ for all $x \in \mathbb{R}$, the equation $\cos x = -2$ cannot have any solution in $\mathbb{R}$.

(v) $f(x) > 0$ for all $-2 \leqslant x < 0$ and $f(x) < 0$ for all $0 \leqslant x \leqslant 2$. If $f(x)$ where continuous this would imply that $f(0) = 0$. But the function is not continuous at $x = 0$ ($f(0^-) = 2$, $f(0^+) = -2$), so there is no solution to the equation $f(x) = 0$ in $[-2,2]$.

(vi) Denoting

$$f(x) = \frac{x^3}{4} - \sin(\pi x) + 3 - \frac{7}{3} = \frac{x^3}{4} - \sin(\pi x) + \frac{2}{3},$$

$f(-2) = -4/3$ and $f(2) = 8/3$, so Bolzano's theorem guarantees at least one zero in $[-2, 2]$.

(vii) Clearly $|\sin x| - \sin x \leqslant 2$, so the equation $|\sin x| - \sin x = 3$ cannot have any solution in $\mathbb{R}$.

**Problem 6.6** If $f(x) = a_{2n+1}x^{2n+1} + a_{2n}x^{2n} + \cdots + a_1 x + a_0$ then, as $x \to \pm\infty$ we have $f(x) \sim a_{2n+1}x^{2n+1}$. Therefore the signs of $f(x)$ for large positive $x$ and large negative $x$ are opposite, so we can apply Bolzano and conclude that $f(x)$ must be zero at least at one point in $\mathbb{R}$.