Midterm Exam II

- No books, notes or electronic devices – except non-programmable pocket calculators – are allowed.
- All answers must be properly justified and the result clearly stated.

Question 1. (*2 marks*)
Consider the function

$$f(x) = (x+1)(x-1)^{2/3} - 3/2 \quad \text{for} \quad x \in \mathbb{R}.$$

(a) Find the relative extrema (if any) of $f(x)$ and discuss the monotonicity of $f(x)$ (1 mark).
(b) Find the inflexion points (if any) of $f(x)$ and discuss the convexity of $f(x)$ (1 mark).

Question 2. (*2.5 marks*)
Consider the function

$$f(x) = \frac{x}{x-1} \quad \text{for} \quad x \in \mathbb{R} \setminus \{1\}.$$

(a) For $f(x)$, give Taylor's polynomial $P_{3,0}(x)$ and the remainder term $R_{3,0}(x)$ (2 marks).
(b) Give the absolute error when approximating $f(0.5)$ by $P_{3,0}(0.5)$ (0.5 marks).

Question 3. (*3 marks*)
(a) Determine

$$F(x) = \int \frac{x^2}{1+x^6} dx.$$

(b) Determine

$$I = \int_0^{2\pi} 2x(\sin^2(x) + \cos^2(x)) dx.$$

(c) Determine the length of the parabola $y = x^2$ in the interval $[0, 2]$.
Remark: (a) is worth 1 mark, (b) is worth 0.5 marks, (c) is worth 1.5 marks.

Question 4. (*2.5 marks*)
(a) An infinite sequence is given by the recurrence relation $x_{k+1} = 1 + \dfrac{1}{x_k}$ with starting value $x_0 = 1$. Discuss whether the sequence converges or diverges. If it converges, give the result. Hint: if a recurrent sequence converges, it fulfills $x_{k+1} = x_k$ when k tends to infinity. (1.5 marks)
(b) Discuss whether

$$\sum_{k=0}^{\infty} 5^{-k}$$

converges or diverges. If it converges, give the result (1 mark).

1) $f(x) = (x+1)(x-1)^{2/3} - \frac{3}{2} \quad , x \in \mathbb{R}$

a) monotonicity / extrema

$$f'(x) = (x+1)\frac{2}{3}(x-1)^{-\frac{1}{3}} + 1 \cdot (x-1)^{2/3}$$

$$= \frac{2(x+1)}{3(x-1)^{\frac{1}{3}}} + \frac{(x-1)^{2/3} \cdot 3(x-1)^{\frac{1}{3}}}{3(x-1)^{\frac{1}{3}}} = \frac{5x-1}{3(x-1)^{\frac{1}{3}}}$$

$$\left.\begin{array}{l} f'(x) \text{ not defined at } x = 1 \\ f'(x) = 0 \quad \text{at } x = \frac{1}{5} \end{array}\right\} \text{critical points}$$

sign of f' = monotonicity

$(-\infty, \frac{1}{5})$	$(\frac{1}{5}, 1)$	$(1, \infty)$
$+$	$-$	$+$

f increases in $(-\infty, \frac{1}{5}) \cup (1, \infty)$ and decreases in $(\frac{1}{5}, 1)$.
At $\left(\frac{1}{5}, \frac{12}{5}\left(\frac{2}{25}\right)^{\frac{1}{3}} - \frac{3}{2}\right)$ there is a relative maximum,
at $(1, -\frac{3}{2})$ there is a relative minimum.

b) convexity / inflexion points

$$f''(x) = \frac{3(x-1)^{\frac{1}{3}} \cdot 5 - (5x-1)(x-1)^{-\frac{2}{3}}}{9(x-1)^{\frac{2}{3}}} = \frac{15(x-1) - (5x-1)}{9(x-1)^{\frac{4}{3}}} =$$

$$= \frac{10x-14}{9(x-1)^{\frac{4}{3}}} \quad ; \quad 9(x-1)^{\frac{4}{3}} > 0 \text{ for all } x \neq 1$$

$$f''(x) = 0 \quad \text{at } x = \frac{14}{10} = \frac{7}{5}$$

sign of f''

$(-\infty, \frac{7}{5})$	$(\frac{7}{5}, \infty)$
$-$	$+$

f is concave in $(-\infty, \frac{7}{5})$ and convex in $(\frac{7}{5}, \infty)$

At $\left(\frac{7}{5}, \frac{12}{5}\left(\frac{2}{5}\right)^{2/3} - \frac{3}{2}\right)$ there is an inflexion point.

2) $f(x) = \dfrac{x}{x-1} \qquad x \in \mathbb{R} \setminus \{1\} \qquad \Rightarrow f(x_0=0) = 0$

a) $f'(x) = \dfrac{(x-1) - x}{(x-1)^2} = \dfrac{-1}{(x-1)^2} \qquad f'(x_0=0) = -1$

$f''(x) = \dfrac{+2(x-1)}{(x-1)^4} = \dfrac{2}{(x-1)^3} \qquad f''(x_0=0) = -2$

$f'''(x) = \dfrac{-2 \cdot 3 \cdot (x-1)^2}{(x-1)^6} = \dfrac{-6}{(x-1)^4} \qquad f'''(x_0=0) = -6$

$f^{(4)}(x) = \dfrac{+6 \cdot 4 \cdot (x-1)^3}{(x-1)^8} = \dfrac{24}{(x-1)^5} \qquad f^{(4)}(c) = \dfrac{24}{(c-1)^5}$

$f(x) = P_{3,0}(x) + R_{3,0}(x)$, with

$P_{3,0}(x) = f(x_0) + \dfrac{f'(x_0)}{1!} x + \dfrac{f''(x_0)}{2!} x^2 + \dfrac{f'''(x_0)}{3!} x^3$

$\underline{\underline{P_{3,0}(x) = -x - x^2 - x^3}}$

$\underline{\underline{R_{3,0}}}(x) = \dfrac{f^{(4)}(c)}{4!} x^4 = \underline{\underline{\dfrac{x^4}{(c-1)^5}}}$

b) $P_{3,0}(0.5) = -0.5 - 0.25 - 0.125 = -0.875$

$f(0.5) = \dfrac{0.5}{0.5-1} = -1$

$E = |P_{3,0}(0.5) - f(0.5)| = \underline{\underline{0.125}} \left(= \frac{1}{8}\right)$

3 a) $F(x) = \int \frac{x^2}{1+x^6} dx$

$$x^3 = u \qquad \frac{du}{dx} = 3x^2 \qquad x^2 dx = \frac{du}{3}$$

$$F(u) = \int \frac{1}{1+u^2} \frac{du}{3} = \frac{1}{3} \int \frac{du}{1+u^2} \overset{\text{LIST}}{=} \frac{1}{3} \arctan u + C$$

$$\Rightarrow F(x) = \underline{\underline{\frac{1}{3} \arctan(x^3) + C}}$$

b)

$$I = \int_0^{2\pi} 2x \underbrace{(\sin^2 x + \cos^2 x)}_{=1} dx = \left[x^2\right]_0^{2\pi} = \underline{\underline{4\pi^2}}$$

c) $l = \int_a^b \sqrt{1+[f'(x)]^2}\, dx$ with $f(x) = x^2$ and $a=0, b=2$

$f'(x) = 2x$

$l = \int_0^2 \sqrt{1+(2x)^2}\, dx$ for convenience $u = 2x; \frac{du}{dx} = 2$

then, the limits change to 0 and 4.

$$l = \int_0^4 \sqrt{1+u^2} \frac{du}{2} = \frac{1}{2} \int_0^4 \sqrt{1+u^2}\, du \overset{\text{LIST}}{=}$$

$$= \frac{1}{2}\left[\frac{1}{2} u\sqrt{1+u^2} + \frac{1}{2} \ln\left(u + \sqrt{1+u^2}\right)\right]_0^4 =$$

$$= \frac{1}{4}\left[4 \cdot \sqrt{17} + \ln(4+\sqrt{17})\right] = \underline{\underline{\sqrt{17} + \frac{1}{4}\ln(4+\sqrt{17})}} \approx 4.647$$

(3 d.p.)

4 a) The first terms of $x_{k+1} = 1 + \frac{1}{x_k}$, $x_k = 1$

are $\{2, 1.5, 1.6667, 1.6, 1.625, 1.6154, \ldots\}$

which suggests a converging sequence with a limit point close to 1.62. We set $x_{k+1} = x_k = x$ and analyse

$$x = 1 + \frac{1}{x} \qquad x^2 - x - 1 = 0$$

$$\left(x - \frac{1}{2}\right)^2 = 1 + \frac{1}{4} = \frac{5}{4}$$

$$x_{1,2} = \frac{1}{2} \pm \sqrt{\frac{5}{4}}.$$ Since $x > 0$, we propose as limit point value $x = \frac{1}{2}(1+\sqrt{5}) \approx 1.62$ (2 d.p.)

b) $$\sum_{k=0}^{\infty} 5^{-k} = \sum_{k=0}^{\infty} \frac{1}{5^k} = \sum_{k=0}^{\infty} \left(\frac{1}{5}\right)^k = 1 + \frac{1}{5} + \frac{1}{5^2} + \cdots$$

is a geometric series with $s = 1$ and $r = \frac{1}{5}$.

Since $\left|\frac{1}{5}\right| < 1$, the series converge to

$$S = \frac{s}{1-r} = \frac{1}{1-\frac{1}{5}} = \frac{5}{4}.$$

The quotient test also shows that the series is convergent.