Problem 1 **(1.8 points)** Find all the real values x and y such that

$$|1-(x+i\,y)| = x+i\,y.$$

For any complex number $z \in \mathbb{C}$, $|z|$ (the modulus of z) is a real number. Then the left side of the given equation is real, and therefore the right side must also be real. This means that $y=0$.

With this in mind, the equation becomes $|1-x| = x$.

So either $1-x=x$ or $-(1-x)=x$.

But the second equation implies $1=0$, so we discard it. From the first equation we get $x=1/2$.

Answer: $x=1/2$, $y=0$.

Problem 2 (1.2 points) Choose any three of the following limits (0.4 points each) and compute their value without using L'Hôspital's Rule.

a) $\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)$ b) $\lim_{x\to\infty}\frac{\sin x}{x^2+1}$

c) $\lim_{x\to 0}\frac{1-\sqrt{1-x^2}}{x^2}$ d) $\lim_{x\to\infty}[\cos(1/x)]^{x^2}$

You can do a fourth limit. If done correctly, it will add up to 0.4 extra points to your grade.

a) Letting $x\to\infty$, $(\sqrt{x^2+x}-x)\to\infty-\infty$.
This is indeterminate because ∞ is not a real number and $\infty-\infty$ is not defined.
We proceed as follows to eliminate the indeterminacy:

$$\lim_{x\to\infty}(\sqrt{x^2+x}-x)\cdot\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}$$

$$=\lim_{x\to\infty}\frac{x^2+x-x^2}{\sqrt{x^2+x}+x}=\lim_{x\to\infty}\frac{x}{\sqrt{x^2+x}+x}=\frac{\infty}{\infty}$$

To handle this indeterminacy $\left(\frac{\infty}{\infty}\right)$, we must factor out and cancel out the dominant term (largest power of x in this case) from the numerator and denominator.

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x} = \lim_{x\to\infty} \frac{1}{\sqrt{1+1/x}+1}$$

$$= \frac{1}{2}.$$

$\lim_{x\to\infty} \frac{1}{x} = 0$

(b) Observe that $\lim_{x\to\infty} \sin x$ does not exist but $-1 \le \sin x \le 1$. Therefore,

$$\frac{-1}{x^2+1} \le \frac{\sin x}{x^2+1} \le \frac{1}{x^2+1}$$

Since $\lim\limits_{x\to\infty} \dfrac{-1}{x^2+1} = \lim\limits_{x\to\infty} \dfrac{1}{x^2+1} = 0$,

by the Sandwich rule, we have

$\lim\limits_{x\to\infty} \dfrac{\sin x}{x^2+1} = 0$.

(c) $\lim\limits_{x\to 0} \dfrac{1-\sqrt{1-x^2}}{x^2} = \dfrac{0}{0}$

This value is undefined, so we must eliminate the factors that vanish (become 0) from the numerator and denominator. We proceed as follows:

$$\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x^2} \cdot \frac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}}$$

$$= \lim_{x\to 0} \frac{1-(1-x^2)}{x^2(1+\sqrt{1-x^2})} = \lim_{x\to 0} \frac{1}{1+\sqrt{1-x^2}} = \frac{1}{2}$$

d) $\lim_{x\to\infty} \left[\cos(1/x)\right]^{x^2}$

One way to make sure that we avoid any indeterminacy (1^∞ in this case) is to rewrite the limit as

$$\lim_{x\to\infty} \left[1+\cos(1/x)-1\right]^{x^2}$$

$$= \lim_{x\to\infty} \left[\left(1+\cos(1/x)-1\right)^{\frac{1}{\cos(1/x)-1}}\right]^{\frac{\cos(1/x)-1}{(1/x)^2}}$$

$$\underset{y=1/x}{=} \lim_{y\to 0^+} \left[\left(1+\cos y-1\right)^{\frac{1}{\cos y-1}}\right]^{\frac{\cos y-1}{y^2}}$$

Since

- $$\lim_{y \to 0^+} (1 + \cos y - 1)^{\frac{1}{\cos y - 1}} \underset{z = \cos y - 1}{=} \lim_{z \to 0^-} (1+z)^{1/z} = e$$

- $$\lim_{y \to 0^+} \frac{\cos y - 1}{y^2} = -\frac{1}{2}$$

(limits from the special list)

Then

$$\lim_{x \to \infty} [\cos(1/x)]^{x^2} = e^{-1/2}$$

Problem 3 **(2 points)** Using the Principle of Induction, prove that for every natural number n, 4 divides $(5^n - 1)$.

In order to obtain full credit in this problem, you must clearly verify the initial case, as well as state the induction hypothesis, the induction step, and where you used the induction hypothesis and the hint.

Hint. A natural number N is divisible by 4 if and only if $N = 4\,k$ for some natural number k (e.g., $0 = 4 \cdot 0$, $4 = 4 \cdot 1$, $24 = 4 \cdot 6$, $124 = 4 \cdot 31$, etc.). Therefore, you need to show that for every $n \geq 0$, there exists a natural number k such that $5^n - 1 = 4\,k$ or, equivalently, $5^n = 4\,k + 1$.

We verify the initial case $(n=0)$:

$5^0 - 1 = 0 = 4 \cdot 0$

That is, $5^0 - 1$ is divisible by 4.

We assume that $5^n - 1$ is divisible by 4 (induction hypothesis). Then $5^n - 1 = 4k$ for some natural number k.

We must show that $5^{n+1} - 1 = 4 \cdot m$ for some natural number m (induction step).

$$5^{n+1}-1 = 5 \cdot 5^n - 1 = 5(4k+1)-1$$

induction hypothesis

$$= 20k+4$$

$$= 4(5k+1)$$

$$= 4m$$

$m = 5k+1$.

and the proof by induction is complete.