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## Midterm Exam I. Page 2 of 2.

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The terms of this examination formulated on page 1 also apply to page 2.

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Problem 4. (*1 mark*) Study the continuity of the following function in  $x = 0$ . If the function is discontinuous, state the type of discontinuity.

$$f(x) = \begin{cases} \frac{e^{-x} - e^x}{x} & \text{for } x \neq 0 \\ 2 & \text{for } x = 0 \end{cases}$$

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Problem 5. (*1.5 marks*) By applying the definition of the derivative, determine the derivative of

$$f(x) = \frac{x}{x+1}$$

in a point  $x = a$ . Give the domains of  $f$  and  $f'$ .

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Problem 6. (*1 mark*) Consider the function

$$f(x) = (x+1)(x-1)^{2/3} - 3/2$$

on the interval  $[0, 3]$ . Study (a) the continuity of  $f$  and (b) the differentiability of  $f$ .

Remark: (a) is worth 0.5 marks, (b) is worth 0.5 marks.

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Problem 7. (*1.5 marks*) Consider the function

$$f(x) = (1-x)^3$$

on the interval  $[-2, 2]$ . Justify the applicability of the mean value theorem and find the value(s) of  $c$  (from the theorem) satisfying the theorem.

$$4. f(x) = \begin{cases} \frac{e^{-x} - e^x}{x} & \text{for } x \neq 0 \\ 2 & \text{for } x = 0 \end{cases}$$

$$\lim_{x \rightarrow 0^\pm} \frac{e^{-x} - e^x}{x} = \left[ \frac{0}{0} \right] \stackrel{\text{L'Hopital}}{=} \lim_{x \rightarrow 0^\pm} \frac{-e^{-x} - e^x}{1} = -2 = L$$

$$f(0) = 2 \neq L$$

$\Rightarrow f$  has a removable/evitable discontinuity at  $x=0$

$$5. f(x) = \frac{x}{x+1} \quad \text{dom}(f) = \mathbb{R} \setminus \{-1\}$$

$$\begin{aligned} f'(a) &\stackrel{\text{def}}{=} \lim_{\Delta x \rightarrow 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{a+\Delta x}{a+\Delta x+1} - \frac{a}{a+1}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left( \frac{(a+\Delta x)(a+1) - a(a+\Delta x+1)}{(a+\Delta x+1)(a+1)} \right) \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left( \frac{a^2 + a + a\Delta x + \Delta x - a^2 - a\Delta x - a}{(a+\Delta x+1)(a+1)} \right) \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left( \frac{\Delta x}{(a+\Delta x+1)(a+1)} \right) = \lim_{\Delta x \rightarrow 0} \frac{1}{(a+\Delta x+1)(a+1)} = \frac{1}{(a+1)^2} \end{aligned}$$

$$\text{dom}(f') = \mathbb{R} \setminus \{-1\} \text{ since } f'(x) = \frac{1}{(x+1)^2}$$

( $f$  is not defined in  $x=-1$ , and hence neither continuous nor differentiable there.)

Comment : there is an equivalent solution using

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \text{ or using } h \text{ instead of } \Delta x.$$

$$6. f(x) = (x+1)(x-1)^{\frac{2}{3}} - \frac{3}{2}$$

a)  $(x-1)^2$ ,  $(x+1)$  and  $(-\frac{3}{2})$  are polynomials of degree 2, 1 and 0 and as such continuous in  $\mathbb{R}$ .  
The root function  $x^{\frac{2}{3}}$  (cube root) is continuous in  $\mathbb{R}$ .  
Sums and products of continuous functions are continuous in their domains (here, in  $\mathbb{R}$ ), and hence in  $[0, 3]$ , as well as the composition  $(x-1)^{\frac{2}{3}}$ .

b)  $f'(x) = (x+1) \cdot \frac{2}{3} (x-1)^{-\frac{1}{3}} + (x-1)^{\frac{2}{3}} \cdot 1$   
 $= \frac{2(x+1)}{3(x-1)^{\frac{1}{3}}} + \frac{(x-1)^{\frac{2}{3}} \cdot 3(x-1)^{\frac{1}{3}}}{3(x-1)^{\frac{1}{3}}} = \frac{5x-1}{3(x-1)^{\frac{1}{3}}}$

$\Rightarrow f$  is not differentiable in  $x=1$  ~~at~~, but in the rest of the interval it is.

7.  $f(x) = (1-x)^3 \quad \text{dom}(f) = [-2, 2] = I$

$f$  is a polynomial (of order 3) and as such continuous and differentiable in  $\mathbb{R}$  (and hence in  $I$ ). Therefore, the MVT can be applied:  $\exists c \in (-2, 2)$  with

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad \text{with } a=-2, b=2$$

$$f'(c) = \frac{(-1)^3 - 3^3}{2+2} = \frac{-1-27}{4} = -7$$

Since  $f'(x) = 3(1-x)^2 \cdot (-1)$ , we solve  
 $-3(1-c)^2 = -7$  or  $(1-c)^2 = \frac{7}{3} \Rightarrow c_{1,2} = 1 \pm \sqrt{\frac{7}{3}}$

Since  $\frac{7}{3} > 1$  and  $\sqrt{\frac{7}{3}} > 1$ ,  $c_1 = 1 + \sqrt{\frac{7}{3}} \notin (-2, 2)$

and  $c_2 = 1 - \sqrt{\frac{7}{3}}$  is the only solution  $\in (-2, 2)$ .