Retake Exam

Mathematical Methods of Bioengineering
Ingenería Biomédica - INGLÉS

21 of June 2019

*The maximum time to make the exam is 3 hours. You are allowed to use a calculator and two sheets with annotations. IMPORTANT: Question **2 A** is only for "alumnos nuevo ingreso" and question **2 B** is only for "alumnos veteranos".*

Problems

1. Consider the surface $z = x^2 - 6x + y^3$.

   (a) (**1 point**) Find the tangent plane at the origin.

   (b) (**1 point**) Find the point/s on the surface, where the tangent plane are parallel to the plane $\pi : 4x - 12y + z = 7$.

SOLUTION

(a) The formula for the tangent plane of a (explicit) function is:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Substituting at the origin, $(a,b) = (0,0)$, we simplify to:

$$z = 0 + f_x(0,0)(x-0) + f_y(0,0)(y-0)$$

Now computing the partial derivative we get $f_x(x,y) = 2x - 6$ and $f_y(x,y) = 3y^2$. So evaluating it at the origin,

$$z = 0 - 6(x-0) + 0(y-0) = -6x \quad (1)$$
$$z = -6x \quad (2)$$

(b) We write the tangent plane formula in the same form as π:

$$\begin{aligned} -f_x(a,b)x - f_y(a,b)y + z &= D, \quad D = f(a,b) - af_x(a,b) - bf_y(a,b) \\ 4x \quad - 12y + \; z &= 7 \end{aligned}$$

Because the planes are parallel we conclude that,

$$4 = -f_x(a,b) = -(2a-6) \to a = 1$$
$$-12 = -f_y(a,b) = -(3b^2) \to b = \pm\sqrt{4} = \pm 2$$

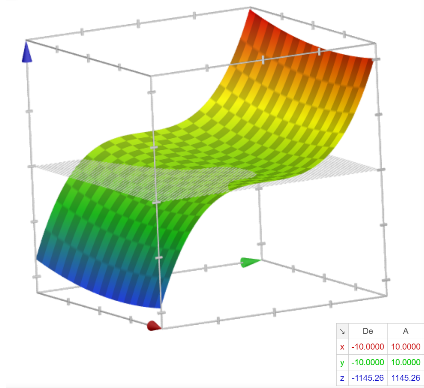

Figure 1: Surface from ex. 1.

2. **A**. Consider the function $f(x, y) = \dfrac{\sin \pi x}{1 + y^2}$.

   (a) (**1 point**) Find the critical points of f.

   (b) (**1 point**) Find the extrema nature of the critical point $(\frac{1}{2}, 0)$.

SOLUTION

(a) We compute the gradient and find when it becomes $\vec{0}$.

$$\nabla f(x, y) = \left(\frac{\pi \cos \pi x}{1 + y^2}, \frac{-2y \sin \pi x}{(1 + y^2)^2} \right)$$

The denominator is always positive so the vector will be zero when the numerator is. For the first component,

$$\pi \cos \pi x = 0 \iff \cos \pi x = 0 \iff x = \frac{1}{2} + k,\ k \in \mathbb{Z}$$

For the second component,

$$-2y \sin \pi x = 0 \iff y = 0 \text{ or } \sin \pi x = 0$$

But $\sin \pi x = \sin \pi(\frac{1}{2} + k) = \pm 1$ so to annul the first and second component of ∇f at the same time, $y = 0$. So, there are infinite critical points of the form:

$$P_k = (1/2 + k, 0),\ k \in \mathbb{Z}$$

(b) We need to compute the Hessian Matrix and apply the criterion. For that we compute the second order derivatives.

- $f_{xx} = \dfrac{-\pi^2 \sin \pi x}{1 + y^2}$
- $f_{xy} = \dfrac{-2\pi y \cos \pi x}{(1 + y^2)^2}$
- $f_{yy} = \dfrac{-2 \sin \pi x (1 + y^2)^2 - 2(1 + y^2) 2y \cdot (-2y \sin \pi x)}{(1 + y^2)^4} = \ldots = \dfrac{\sin \pi x [-2 + 6y^2]}{(1 + y^2)^3}$

Then,

$$H_f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} =$$

$$\begin{bmatrix} \dfrac{-\pi^2 \sin \pi x}{1+y^2} & \dfrac{-2\pi y \cos \pi x}{(1+y^2)^2} \\ \dfrac{-2\pi y \cos \pi x}{(1+y^2)^2} & \dfrac{\sin \pi x[-2+6y^2]}{(1+y^2)^3} \end{bmatrix}.$$

At the given point,

$$H_f(1/2, 0) = \begin{pmatrix} -\pi^2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Because $d_1 = -\pi^2 < 0$ and $d_2 = |H| = -\pi^2(-2) = 2\pi^2 > 0$ we conclude that is a local maximum.

B. A engineering is working with two mechanical arms with movements in a plane. To make a labor minimising the effort, he found that the optimal trajectories of the arm hands are $\mathbf{m_1}(t) = (t^2-2, \frac{t^2}{2}-1)$ and $\mathbf{m_2}(t) = (t, 5-t^2)$, where t represents time measured in seconds. Before running the experiment, he simulated the trajectories and found out that collide.

(a) (**1 point**) When and where do the arms collide?

(b) (**1 point**) What is the angle formed by the paths of the arms at the collision point?

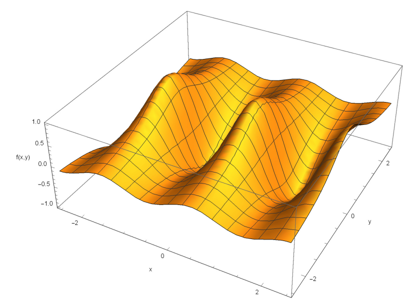

Figure 2: Function $f(x,y) = \dfrac{\sin \pi x}{1+y^2}$ from 2 **A**.

SOLUTION

(a) We have to intersect both paths to find the collision point, $\mathbf{m_1}(t) = \mathbf{m_2}(t)$. This yields in the following equations:

$$\begin{cases} t &= t^2 - 2 \quad \to t^2 - t - 2 = 0 \to t = 2 \text{ or } -1 \\ 5 - t^2 &= \frac{t^2}{2} - 1 \quad \to 3t^2 = 12 \to t = \pm 2 \end{cases}$$

So the only solution for both equations is $t = 2\ s$. Then the collision point is $\mathbf{m_1}(2) = \mathbf{m_2}(2) = (2, 1)$.

(b) The direction of the path is given by the velocity vector (vector tangent to the path). At the collision point, this vectors are:

$$\begin{cases} \mathbf{m_1}'(t) = (2t, t) \quad \to \mathbf{m_1}'(2) = (4, 2) \\ \mathbf{m_2}'(t) = (1, -2t) \quad \to \mathbf{m_2}'(2) = (1, -4) \end{cases}$$

On the other hand, the dot product between two vectors is $< a, b >= \|a\|\|b\| \cos\theta$. So,

$$\cos\theta = \frac{< a, b >}{\|a\|\|b\|} = \frac{4-8}{\sqrt{20}\sqrt{17}} = -0.2169 \to \theta \approx 1.78 \text{ rad} \approx 102.52^\circ$$

3. (**2 points**) Suppose you are working doing artificial organs/tissues and you have a method to print in 3D materials with varying density. For a first test you decide to print a tissues that is modelled as the solid bounden by the surface $z = 1 - x^2$ and the planes $z = 0$, $y = 1$ and $y = -1$, as shown in the next figure.

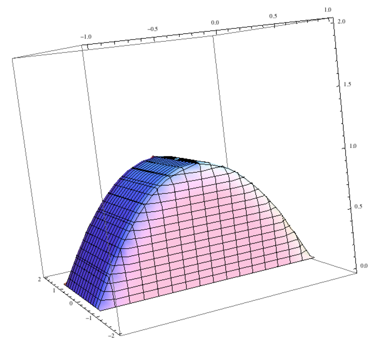

Figure 3: Artificial tissue.

Suppose also that the density is varying according to the function $g(x, y, z) = z(x + 2)$. Compute the total mass of the tissue.

SOLUTION

We have to integrate the density function over the tissue region to get the total mass. The tissue is a region of type 1 in $\mathbb{R}^3$. It is delimited in the $xy-$plane (projection), as the statement says, by $-1 \leq y \leq 1$ and x limits have to be found. Because, when making $z = 0$ we get that $0 = 1 - x^2 \to x = \pm 1$, x is also in $[-1, 1]$.

So the integration region is,

$$W = \{(x, y, z) \in \mathbb{R}^3 : -1 \leq x \leq 1, -1 \leq y \leq 1,\ 0 \leq z \leq 1 - x^2\}$$

Now we compute the mass,

$$\begin{aligned} M &= \int\int\int_W g\ dW = \int_{-1}^{1}\int_{-1}^{1}\Big(\int_0^{1-x^2} z(x+2)\ dz\Big)dydx \\ &= \int_{-1}^{1}\int_{-1}^{1} \frac{(1-x^2)^2}{2}(x+2)\ dydx = ... = \frac{32}{15} \end{aligned}$$

4. (**2 points**) Evaluate $\oint_C (x^4y^5 - 2y)dx + (3x + x^5y^4)dy$, where C is the oriented curve pictured below.

SOLUTION

Because is a closed curve in a vector field of class C^1 we can apply the Green theorem. We note that the curve is oriented on the opposite sense of the Green theorem, so we have to add a minus in the equality.

We see that $M(x, y) = x^4y^5 - 2y$ and $N(x, y) = 3x + x^5y^4$, so $M_y = 5x^4y^4 - 2$ and $N_x = 3 + 5x^4y^4$. Let D be the region enclosed by the closed curve C. Then,

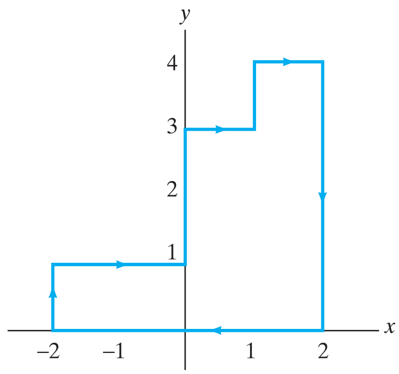

Figure 4: Oriented curve C.

$$\oint Mdx + Ndy = -\int\int_D (N_x - M_y)\ dxdy = -\int\int_D (3 + 5x^4y^4) - (5x^4y^4 - 2)\ dxdy =$$
$$-\int\int_D 5\ dxdy = -5 \cdot \text{Area(D)} = -5(2+3+4) = -45$$

5. Consider the curve $r(t) = (\cos t, \sin t, (a\cos t + b\sin t))$ with $0 \leq t \leq 2\pi$, where $a, b \in \mathbb{R}$ are constants.

   (a) (**1 point**) Compute the work done by the vector field $\mathbf{F} = (y, z - x, -y)$ in a particle moving along r.

   (b) (**1 point**) Compute the values of a, b such that the work done is null. Does this mean that the vector field is conservative? Reasonate the answer.

SOLUTION

(a) The work is,

$$W = \int_r \mathbf{F}d\mathbf{s} = \int_0^{2\pi} F(r(t)) \cdot r'(t)dt =$$
$$\int_0^{2\pi} (\sin t, a\cos t + b\sin t - \cos t, -\sin t) \cdot (-\sin t, \cos t, -a\sin t + b\cos t)\ dt =$$
$$\int_0^{2\pi} -\sin^2 t + a\cos^2 t + b\sin t\cos t - \cos^2 t + a\sin^2 t - b\sin t\cos t\ dt =$$
$$\int_0^{2\pi} a(\cos^2 t + \sin^2 t) - 1\ dt = \int_0^{2\pi} a - 1\ dt = 2\pi(a-1)$$

(b) $W = 0 \iff a = 1,\ b \in \mathbb{R}$. It just mean that the work is zero over that path. To check that is conservative we may calculate the rotational, $\text{rot}(\mathbf{F}) = (-2, 0, -2)$. Because the rotational is not zero the vector field is not conservative.