1.

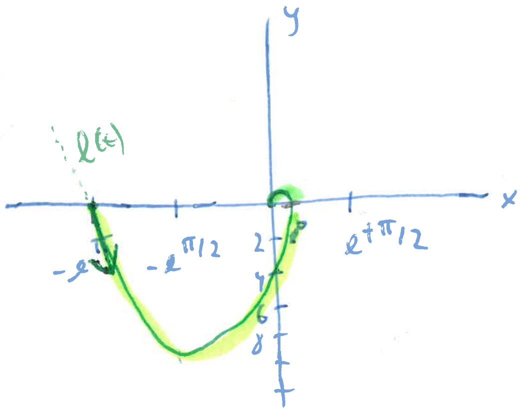
Final Exam Solution

a) $P = (-e^{\pi}, 0) \rightarrow \begin{cases} x = e^{-t}\cos(t) = -e^{+\pi} \\ y = e^{-t}\sin(t) = 0 \end{cases}$ $\leftrightarrow t = -\pi$; $\leftrightarrow t = 0 + k\cdot\pi, \ k \in \mathbb{Z}$

$\vec{c}'(t) = \left[e^{-t}(-\cos(t) - \sin(t)), \ e^{-t}(\cos(t) - \sin(t))\right]$

$\vec{c}'(-\pi) = e^{\pi}\cdot(1, -1)$

Tangent line: $\vec{\ell}(t) = (-e^{\pi}, 0) + (t+\pi)\cdot(1,-1)\, e^{\pi}$



b) $\|\vec{c}'(t)\| = e^{-t}\sqrt{(-c_t - s_t)^2 + (c_t - s_t)^2} =$

$= e^{-t}\sqrt{2}$

$\vec{c}(0) = (1, 0)$ so:

$\rightarrow L_1 = \int_{-\pi}^{0} \sqrt{2}\, e^{-t}\, dt = -\sqrt{2}\, e^{-t}\Big|_{-\pi}^{0} = (e^{\pi} - 1)\sqrt{2}.$ 31,312

$\rightarrow L_2 = \int_0^{\infty} \sqrt{2}\, e^{-t}\, dt = -\sqrt{2}\, e^{-t}\Big|_0^{\infty} = \sqrt{2}.$

$L_1 > L_2.$

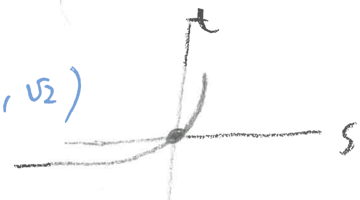
c) $T(t) = \dfrac{\vec{c}'(t)}{\|\vec{c}'(t)\|} = \dfrac{e^{-t}(-c_t - s_t, \ c_t - s_t)}{\sqrt{2}\, e^{-t}} = \dfrac{1}{\sqrt{2}}(-\cos(t) - \sin(t), \ \cos(t) - \sin(t))$

$s(t) = \int_0^t \|\vec{c}(\tau)\|\, d\tau = \sqrt{2} - \sqrt{2}\, e^{-t}$

$\dfrac{s}{\sqrt{2}} = 1 - e^{-t}$

$e^{-t} = 1 - s/\sqrt{2}$

$t = -\ln(1 - s/\sqrt{2}), \quad s \in (-\infty, \sqrt{2})$



$\vec{c}(s) = \dfrac{1}{1 - s/\sqrt{2}}\left[\cos\left(\ln\left(\dfrac{1}{1 - s\sqrt{2}}\right)\right), \ \sin\left(\ln\left(\dfrac{1}{1 - s/\sqrt{2}}\right)\right)\right]$

with $s \in (-\infty, \sqrt{2})$. the speed is always unitory with the arclength parametrization.

d) we will use the formula $\kappa(t) = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|^2}$

$\vec{c}'(t) = e^{-t}[-c_t - s_t, c_t - s_t]$

$\vec{c}''(t) = e^{-t}[2s_t, -2c_t]$

$\vec{c}'(t) \times \vec{c}''(t) = \underbrace{[e^{-2t}(-2c_t(-c_t - s_t)) - e^{-2t}(2s_t(c_t - s_t))]}_{2e^{-2t}[c_t^2 + \cancel{s_t c_t} + s_t^2 - \cancel{s_t c_t}]} \cdot \vec{k}$

$= 2e^{-2t}$

$\|\vec{v}\| = \|\vec{c}'(t)\| = \sqrt{2}\, e^{-t}$

$\kappa(t) = \frac{|2e^{-2t}|}{(\sqrt{2}\, e^{-t})^3} = \frac{\cancel{2} e^{-2t}}{\cancel{2}\sqrt{2}\, e^{-3t}} = \frac{e^t}{\sqrt{2}}$.

The curvature* of a circle is $\frac{1}{r}$ where r is the radius of the circle. so,

$\kappa(t) \overset{!}{=} \frac{1}{1} \Leftrightarrow \frac{e^t}{\sqrt{2}} = 1 \Rightarrow e^t = \sqrt{2}$

$t = \ln(\sqrt{2}) \doteq 0.3466$

the curvature is maximum when $t \to \infty$ and minimum when $t \to -\infty$.

*can be found even faster by the original intrinsic formula:

$\kappa(t) = \frac{\|T'(t)\|}{s'(t)}$

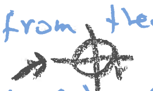
a)

- $F(x,y) = (0, x^2 y)$ is picture 3 since is the only vector fied having first component equal to 0.

- $F(x,y) = (x^2 y, 0)$ is picture 1 " " " second component equal to 0.

- $F(x,y) = (-y-x, x)$ is picture 3. For example, we see that $F(x,y)$ is $\perp$ to the position vector only at points with $x=0$:

$$\langle (x,y), (-y-x, x) \rangle = -x^2$$

- $F(x,y) = (-y, x)$ is picture 4 since we know from theory  that paths of the form $c(t) = (r\cos(t), r\sin(t))$ are flow lines ($F(c(t)) = c'(t)$) of F.

b)

$$\begin{cases} x'(t) = 1 \\ y'(t) = -3y \\ z'(t) = z^3 \end{cases}$$

$x'(t) = 1 \to x(t) = t + k_1$

$y'(t) = -3y \to y(t) = k_2 e^{-3t}$

$z'(t) = z^3 \to \frac{dz}{dt} = z^3 \to \frac{1}{z^3}\, dz = dt$

$$\int z^{-3}\, dz = \int dt$$

$$\frac{z^{-2}}{-2} = t + k_3$$

$$z^2 = \frac{1}{-2(t+k_3)}$$

$$z = \frac{1}{\sqrt{-2(t+k_3)}}$$

we take the + root since $z(0) = 7 > 0$

So $\vec{c}(t) = \left(t+k_1,\ k_2 e^{-3t},\ \frac{1}{\sqrt{-2(t+k_3)}} \right)$

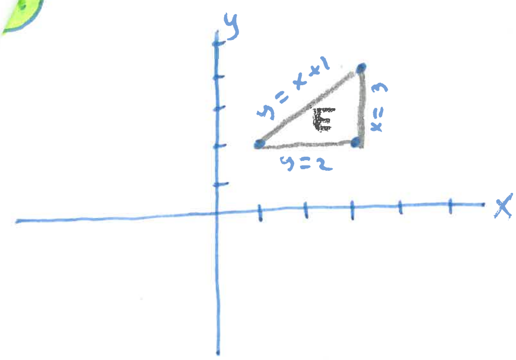
$c(0) = (3, 5, 7)$ So,

$3 = k_1$

$5 = k_2 e^0 = k_2$

$7 = \frac{1}{\sqrt{-2k_3}} \to 49 = \frac{1}{-2k_3}$

$$k_3 = \frac{1}{-98}$$



a) $E = \{(x,y) : 1 \le x \le 3,\ 2 \le y \le x+1\}$

b) $E = \{(x,y) : 2 \le y \le 4,\ y-1 \le x \le 3\}$

Yes, is of type III since is type I and II.

c) $M = \iint_E 24x\, dA = \int_1^3 \int_2^{x+1} 24x\, dy\, dx =$

$= \int_1^3 24x \cdot y \Big|_2^{x+1} dx = \int_1^3 24x(x+1-2)\, dx =$

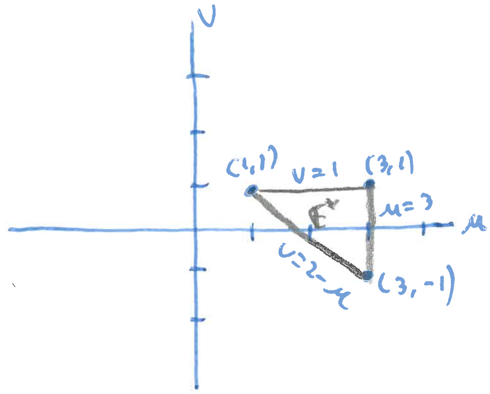
$= 24 \int_1^3 x^2 + x - 2x\, dx = \underbrace{24 \cdot \frac{x^3}{3} - \frac{x^2}{2}}\Big|_1^3$

$= 24\left[9 - \frac{9}{2} - \left(\frac{1}{3} - \frac{1}{2}\right)\right] =$

$= 24\left[\frac{9}{2} - \left(-\frac{1}{6}\right)\right] = \cancel{24}\left(\frac{27+1}{\cancel{6}}\right)$

$= 4 \cdot 28 = 112$ g $= 0{,}112$ kg

d) +0,5



$x = u \rightarrow y - x = v \rightarrow \underline{v = 1}$

$\rightarrow x = u \rightarrow \underline{u = 3}$

$y = u + v \rightarrow y = u + v \rightarrow 2 = u + v$

$\underline{v = 2 - u}$

$$J = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \rightarrow \text{abs}(|J|) = 1$$

$M = \iint_E 24x\, dA = \iint_{E^*} 24u\, dA = \int_1^3 \int_{2-u}^1 24u\, dv\, du =$

$= 24 \int_1^3 u \cdot v \Big|_{2-u}^1 du = 24 \int_1^3 u \cdot (1 - (2-u))\, du =$

$= 24 \int_1^3 -u + u^2\, du = \underbrace{24 \quad \frac{u^3}{3} - \frac{u^2}{2}}\Big|_1^3 = 112$ g

4.

a) C_1 is centred at $(3,0)$ and have radius 3 cm. We need only half of the circumference corresponding to $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ ($-90°$, $90°$).

$$C_1 = (3 + 3\cos(t), 3\sin(t)), \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$
$$C_3 = (-3 + 3\cos(t), 3\sin(t)), \quad \frac{\pi}{2} \le t \le \frac{3\pi}{2}$$

C_2 is the line $y=3$ between the points $(-3,3)$ and $(3,3)$.
C_4 is the line $y=-3$ " " $(-3,-3)$ and $(3,-3)$.

So a parametrization could be:

$$C_2 = (-t, 3), \quad -3 \le t \le 3$$
$$C_4 = (t, 3), \quad -3 \le t \le 3$$

b) We must compute the area under the graph $f(x,y) = 50 - x^2$ restricted to base curve governed by the parametrizations C_1, C_2, C_3, C_4.

$$A = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \int_{C_3} f\,ds + \int_{C_4} f\,ds$$

Because symmetry and to simplify computations we use that $\int_{C_1} f\,ds = \int_{C_3} f\,ds$ and $\int_{C_2} f\,ds = \int_{C_4} f\,ds$. So,

$$A = 2\int_{C_1} f\,ds + 2\int_{C_2} f\,ds$$

- $\int_{C_1} f\,ds = \int_{-\frac{\pi}{2}}^{\pi/2} [50 - (3+3\cos(t))^2] \cdot \underbrace{3}_{\|C_1'(t)\|}\,dt = \int_{-\frac{\pi}{2}}^{\pi/2} 123 - 54\cos(t) - 27\cos^2(t)\,dt$

$$= \frac{219\pi}{2} - 108 \text{ cm}^2$$

- $\int_{C_2} f\,ds = \int_{-3}^{3} [50 - (-t)^2] \cdot \underbrace{1}_{\|C_2'(t)\|}\,dt = 50t - \frac{t^3}{3}\Big|_{-3}^{3} = \ldots = 282 \text{ cm}^2$

So, $A = 2\left(\frac{219}{2}\pi - 108\right) + 2(282) = 219\pi + 348 \text{ cm}^2 \simeq 1036 \text{ cm}^2$

5.

we shall use the Green's theorem with the vector Field.

$$F_1 = (\overset{M}{-y}, \overset{N}{x}) \quad \text{or} \quad F_2 = (\overset{M}{0}, \overset{N}{x}).$$

$$\text{Area} = \frac{1}{2}\oint_\gamma F_1 \, d\vec{s} = \frac{1}{2}\oint_\gamma -y\,dx + x\,dy = \frac{1}{2}\iint_M \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dx\,dy$$

$$= \frac{1}{2}\iint_M 2\,dx\,dy = \text{Area of } M$$

$$\frac{1}{2}\int_0^{2\pi} (-S_t - C_{4t}, 5C_t)\cdot(-5S_t, C_t - 4S_{4t})\,dt =$$

$$= \frac{1}{2}\int_0^{2\pi} 5S_t^2 + 5S_t\cdot C_{4t} + 5C_t^2 - 20\,C_t\cdot S_{4t}\,dt$$

odd function $\to \int_0^{2\pi} \ldots = 0$

$$= \frac{1}{2}\int_0^{2\pi} 5 + 5\,S_t\cdot C_{4t} - 20\,C_t\cdot S_{4t}\,dt = \frac{1}{2}\int_0^{2\pi} 5 - \frac{5}{2}S_{3t} + \frac{5}{2}S_{5t} - \frac{20}{2}S_{3t} - \frac{20}{2}S_{5t}\,dt$$

$$S_t\cdot C_{4t} = \frac{1}{2}S_{-3t} + \frac{1}{2}S_{5t}$$

$$S_{4t}\cdot C_t = \frac{1}{2}S_{3t} + \frac{1}{2}S_{5t}$$

$$= \frac{1}{2}\int_0^{2\pi} 5 \underbrace{- \frac{25}{2}S_{3t} - \frac{15}{2}S_{5t}}\,dt$$

$$= 5\pi + \int_0^{2\pi}(-)\,dt = 5\pi\ cm^2$$

0

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If we were to use $F = (0, x)$

$$A = \int_0^{2\pi}(0, 5C_t)\cdot(-5S_t, C_t - 4S_{4t})\,dt =$$

$$= \int_0^{2\pi} 5C_t^2\,dt - 20\int_0^{2\pi} S_{4t}\cdot C_t\,dt = 5\pi\ cm^2$$

0