Chapter 4 – Line or surface integrals.

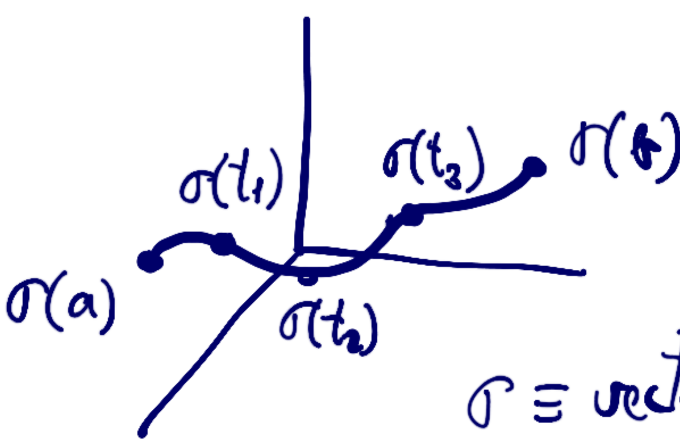
Path/line integrals

Generalisation of 1D integrals

Necessary elements:

Path or trajectory

$$\sigma : [a, b] \longrightarrow \mathbb{R}^N$$
$$t \longrightarrow \sigma(t) \in \mathbb{R}^N$$



$\sigma \equiv$ continuous function (stepwise cont.)

$\sigma \equiv$ vector function of real values.

$\sigma : \mathbb{R} \longrightarrow \mathbb{R}^N$

Image of $\sigma \equiv$ curve.

$$\mathrm{Im}\,\sigma = \sigma([a,b])$$

Different ways of representing σ:

- Most common and useful (in this chapter)

  Parametric equations

Example: $x+y=3$: Implicit equation

$t \equiv$ parameter $\left\{ \begin{array}{l} x = t \\ y = 3-t \end{array} \right\}$: Parametric equation

$y = 3 - x$: Explicit equation

for any $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$: Vector equation.

In general if σ is a trajectory in $\mathbb{R}^3$

$$\sigma(t) = \Big(\sigma_1(t), \sigma_2(t), \sigma_3(t) \Big)$$

for any position $r = (x, y, z)$ we find

$$\left.\begin{aligned} x &= \sigma_1(t) \\ y &= \sigma_2(t) \\ z &= \sigma_3(t) \end{aligned}\right\}$$

: Parametric equations. for the path σ

$$\sigma : [a, b] \longrightarrow \mathbb{R}^N$$

but $N = 2, 3$

Definition

$\sigma : [a,b] \longrightarrow \mathbb{R}^n$ is a continuous path whose curve is given by

$$\boxed{\operatorname{Im}\sigma = \sigma\big([a,b]\big)}$$

We say that σ is smooth if it is differentiable

$$\sigma \in C^1 \text{ or } \sigma'(t) = \big(\sigma_1'(t), \dots, \sigma_n'(t)\big) \text{ cont.}$$

Then, we say that σ is a trajectory.

Definition

Tangent line to a trajectory σ at the point $\sigma(t_0)$

$$S(\lambda) = \sigma(t_0) + \lambda\, \underbrace{\sigma'(t_0)}_{\in \mathbb{R}^n} \qquad \lambda = \text{parameter.}$$

$$t_0 \in [a,b].$$

Remark

In $\mathbb{R}^3$ we can assume the ratio of change in every direction, between two points.

$$t, t_0 \in [a, b]$$

so that

$$\left.\begin{aligned} &\frac{\sigma_1(t) - \sigma_1(t_0)}{t - t_0} \sim \sigma_1'(t_0) \\ &\frac{\sigma_2(t) - \sigma_2(t_0)}{t - t_0} \sim \sigma_2'(t_0) \\ &\frac{\sigma_3(t) - \sigma_3(t_0)}{t - t_0} \sim \sigma_3'(t_0) \end{aligned}\right\}$$

$$\sigma'(t_0) = \left(\sigma_1'(t_0), \sigma_2'(t_0), \sigma_3'(t_0)\right)$$

At the limit

$$\lambda = \frac{\sigma_1(t) - \sigma_1(t_0)}{\sigma_1'(t_0)} = \frac{\sigma_2(t) - \sigma_2(t_0)}{\sigma_2'(t_0)} = \frac{\sigma_3(t) - \sigma_3(t_0)}{\sigma_3'(t_0)}$$

$$\Downarrow$$

$$S(\lambda) = \underbrace{\sigma(t_0)}_{\in \mathbb{R}^n} + \lambda\, \sigma'(t_0)$$

Parametric equation

$$\sigma(t_0) = \Big(\sigma_1(t_0), \sigma_2(t_0), \sigma_3(t_0) \Big)$$

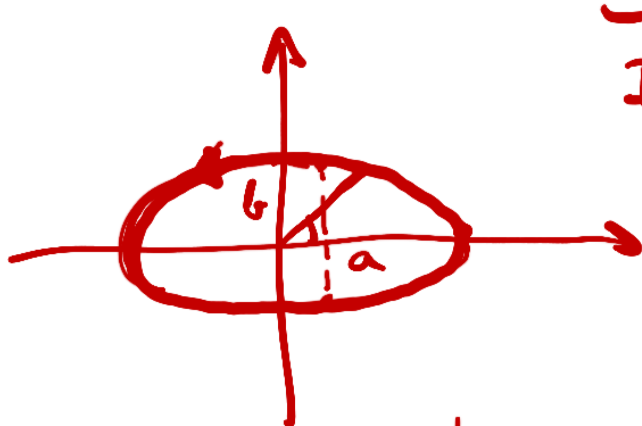
$\sigma_i(t_0) \in \mathbb{R}$.

- Remember.

$\sigma'(t) \equiv$ velocity

$\|\sigma'(t)\| \equiv$ speed

Example: Ellipse $\{ \underbrace{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}_{\text{Implicit equation}}, (x,y)\in\mathbb{R}^2 \}$



To find a trajectory for that curve:

Parametrise the curve.

a) $\left.\begin{array}{l} x=a\cos\theta \\ y=b\sin\theta \end{array}\right\} : \sigma$, $\theta \equiv$ parameter.

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$

$$\sigma(\theta)=\left(\underbrace{a\cos\theta}_{x}, \underbrace{b\sin\theta}_{y}\right)$$

$\theta\in[0,2\pi]$

$$\sigma : [0, 2\pi] \longrightarrow \mathbb{R}^2$$
$$\theta \longrightarrow \sigma(\theta) = (a\cos\theta, b\sin\theta)$$

b) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Divide the ellipse into two different branches.

use explicit formula

$$y = b\sqrt{1 - \frac{x^2}{a^2}} \qquad y = -b\sqrt{1 - \frac{x^2}{a^2}}$$

If
$$\sigma_1 \begin{cases} x = t \\ y = b\sqrt{1 - \frac{t^2}{a^2}} \end{cases}, \quad \sigma_2 : \begin{cases} x = t \\ y = -b\sqrt{1 - \frac{t^2}{a^2}} \end{cases}$$

Example: $x+y+z=1$

Parametrise the plane:

$$\left.\begin{aligned} x &= t \\ y &= s \\ z &= 1-t-s \end{aligned}\right\}$$ Parametric equations.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{P} + t\underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{v_1} + s\underbrace{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}_{v_2}$$ Vector equations

$X \in \mathbb{R}^3$

$X = P + \operatorname{span}\{v_1, v_2\}$

Curves

$\sigma : [a,b] \longrightarrow \mathbb{R}^n$ trajectory

$\operatorname{Im} \sigma = \sigma([a,b])$ curve.

Definition

We say that:

a) $\sigma([a,b])$ is a simple curve if

σ is one-to-one,

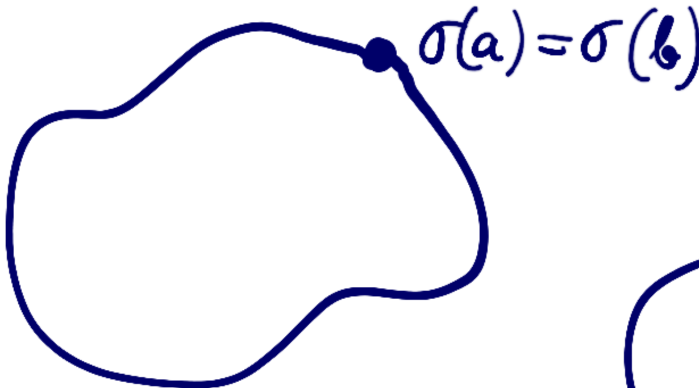
$\sigma(t_0) \neq \sigma(t_1)$ for any $t_0, t_1 \in [a,b]$

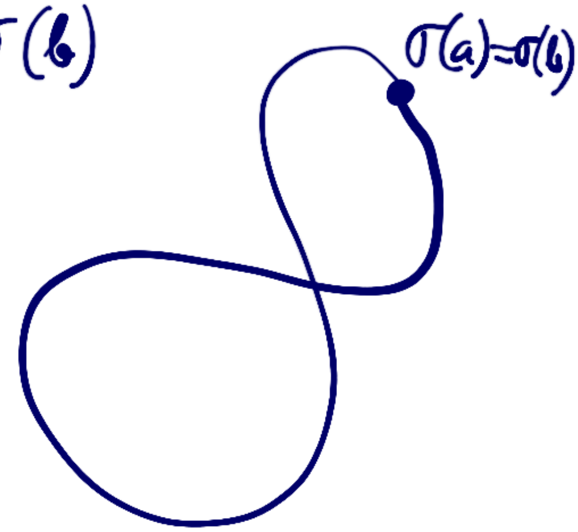
$\sigma(b) \equiv$ end point

$\sigma(a) \equiv$ start point

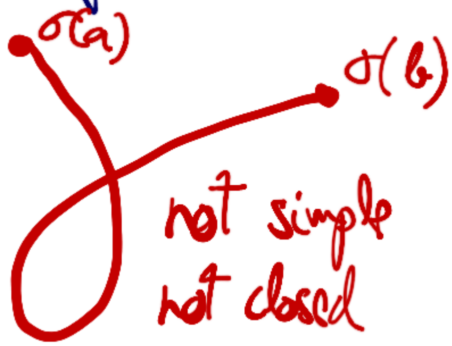
b) $\sigma([a,b])$ is a closed curve if

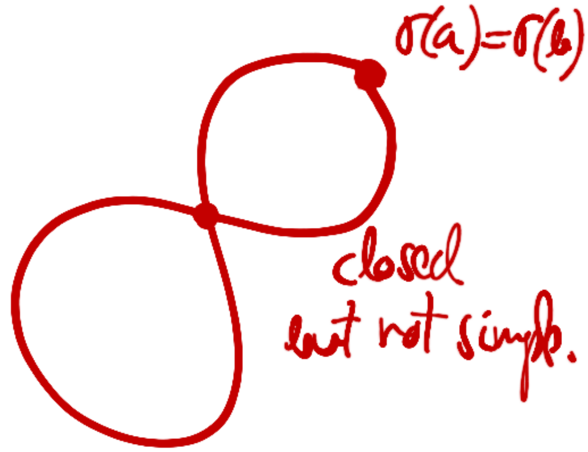
$$\sigma(a) = \sigma(b)$$





c) $\sigma([a,b])$ is a simple and closed curve if it is closed and one-to-one of any t apart from a and b

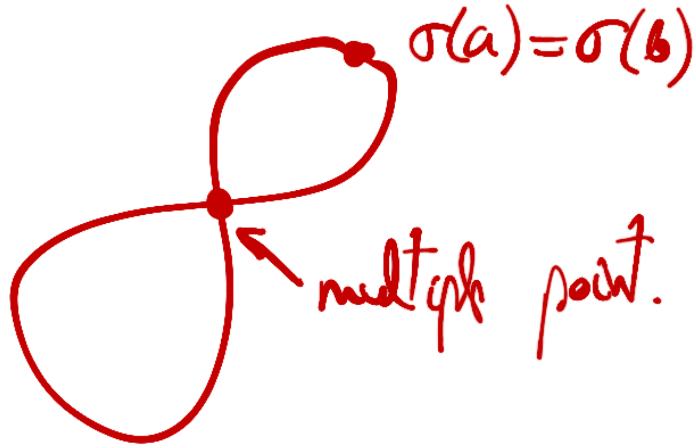




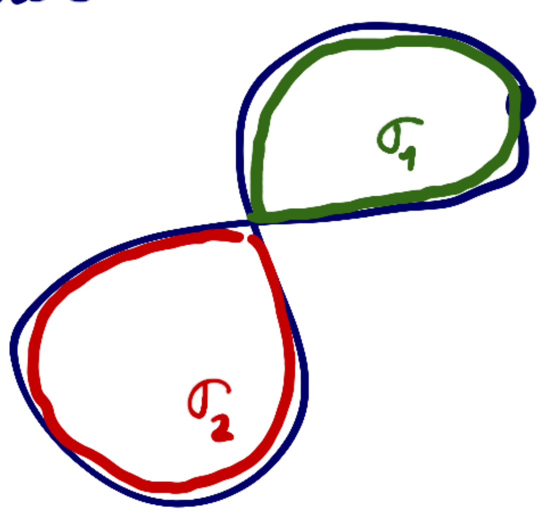
Remark

If $\sigma(t_0) = \sigma(t_1)$ for any $t_0, t_1 \in (a, b)$

then we have a <u>multiple point</u>



Very important concept in integration along trajectories !!



Definition

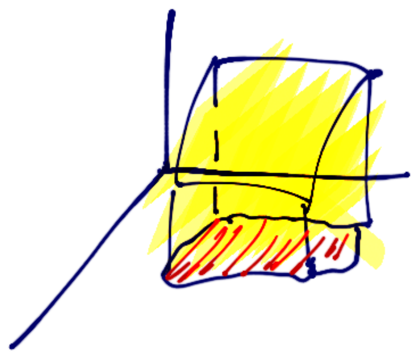
$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ scalar functions (temperature, density)

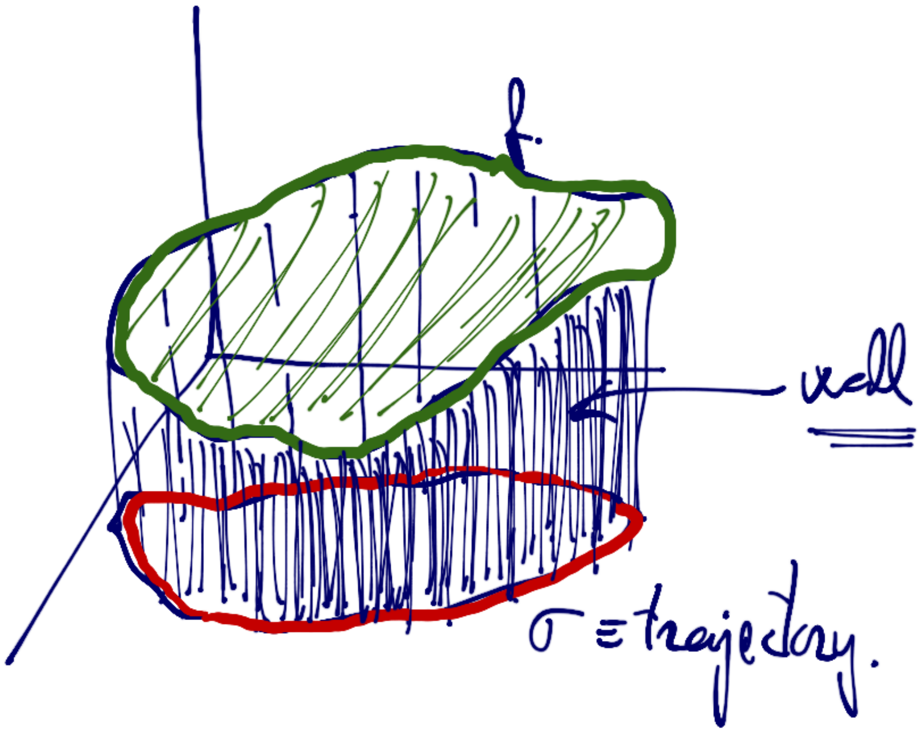
$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ vector functions/fields
(velocity, force, etc)

Remark

Change the perspective:

We don't focus on integrals giving volumes or areas below a function in a region





We compute how that function behaves along that trajectory, we create a wall and compute the area of that wall.

Path integrals for scalar functions.

$\sigma : [a,b] \longrightarrow \mathbb{R}^n$ trajectory.

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$ scalar function.

The idea is to compute

$\int_\sigma f \equiv$ average of the density along that trajectory

Definition

$\sigma : [a,b] \longrightarrow \mathbb{R}^n$ a trajectory of class C^1
$\quad t \longrightarrow \sigma(t)$ (stepwise C^1)

and $f : \underbrace{\sigma([a,b])}_{\text{curve or image of } \sigma} \subset \mathbb{R}^n \longrightarrow \mathbb{R}$

we move every point on the curve by f.

Then, the integral of f along the trajectory σ is given by

$$\boxed{\int_\sigma f = \int_a^b f(\sigma(t)) \cdot \|\sigma'(t)\| \, dt}$$

$$= \int_a^b f(\sigma_1(t), \dots \sigma_n(t)) \cdot \sqrt{(\sigma_1'(t))^2 + \dots + (\sigma_n'(t))^2} \, dt$$

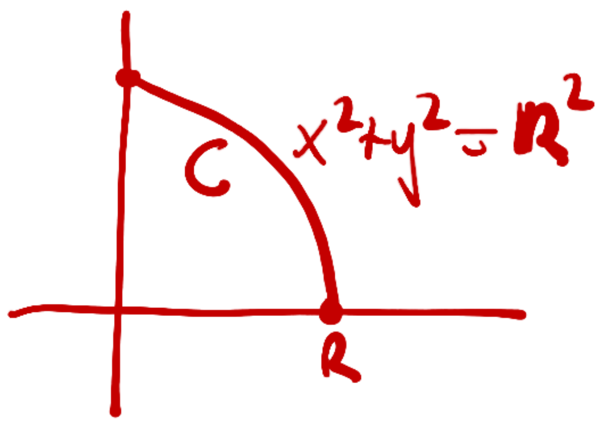
Notation

$$\int_\sigma f \quad \text{or} \quad \int_\sigma f(x_1, \dots x_n)\, ds$$

$ds \equiv$ differential movement along the path.

Problem ① i)
4.1

$f(x,y) = 2xy^2$ along $x^2+y^2=R^2$ in the first quadrant.



$C = \{(x,y) \in \mathbb{R}^2,\ x^2+y^2=R^2,\ x,y \geq 0\}$

$$\int_C f = \int_a^b f(\sigma(t)) \cdot \|\sigma'(t)\|\, dt.$$

First, we parametrise the curve.

$$\sigma : \left.\begin{array}{l} x = R\cos t \\ y = R\sin t \end{array}\right\} \quad t \in [0, \pi/2]$$

$$\sigma : \left[0, \frac{\pi}{2}\right] \longrightarrow \mathbb{R}^2$$

$$t \longrightarrow \sigma(t) = \left(R\cos t,\ R\sin t\right)$$

$$\sigma'(t) = R\left(-\sin t,\ \cos t\right)$$

$$\|\sigma'(t)\| = R\sqrt{\sin^2 t + \cos^2 t} = R$$

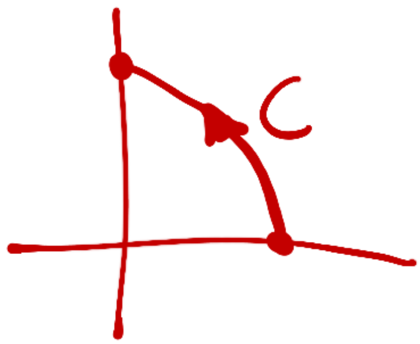
$$f(\sigma(t)) = 2\,\underbrace{R\cos t}_{x}\,\underbrace{R^2\sin^2 t}_{y^2} = \underline{2R^3}\cos t\sin^2 t$$

$$\int_C f = \int_0^{\pi/2} \underbrace{2R^3\cos t\sin^2 t}_{f(\sigma(t))} \cdot \underbrace{R}_{\|\sigma'(t)\|}\,dt = 2R^4\int_0^{\pi/2} \cos t\sin^2 t\,dt$$

$$= 2R^4 \frac{\sin^3 t}{3}\Big]_0^{\pi/2} = \frac{2R^4}{3}$$

$t \in [0, \pi/2]$

$t=0$ start point
$t=\frac{\pi}{2}$ end point



If $t=0$ end point
$t=\frac{\pi}{2}$ start point

$$\int_{C_1} f = -\int_C f$$

:

<u>Definition</u> - length of a curve.

It will be a line integral along a curve with $f(x_1 \ldots x_N) = 1$

$$L = \int_\sigma 1 = \int_a^b \|\sigma'(t)\| \, dt$$

Area of a region D

$$A = \iint_D dx\,dy$$

<u>Definition</u> - Mean value of f along a trajectory

$$\frac{1}{L(\sigma)}\int_\sigma f = \frac{1}{L(\sigma)}\int_a^b f(\sigma(t))\cdot \underbrace{\|\sigma'(t)\|\,dt}_{\text{differential of arc}}$$

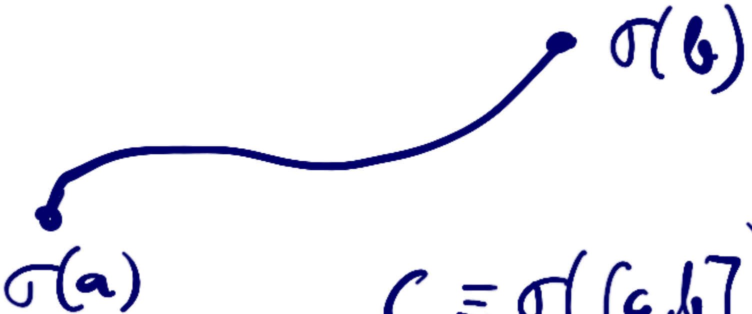
$$\boxed{ds = \|\sigma'(t)\|\,dt}$$ modulus

$$s = x\,i + y\,j + z\cdot k$$

$$\left.\begin{aligned} dx &= \sigma_1'(t)\,dt \\ dy &= \sigma_2'(t)\,dt \\ dz &= \sigma_3'(t)\,dt \end{aligned}\right\}$$ on the direction of the tangent vector.

Geometric interpretation

Assume $\sigma : [a,b] \to \mathbb{R}^N$ trajectory



$C \equiv \sigma([a,b])$

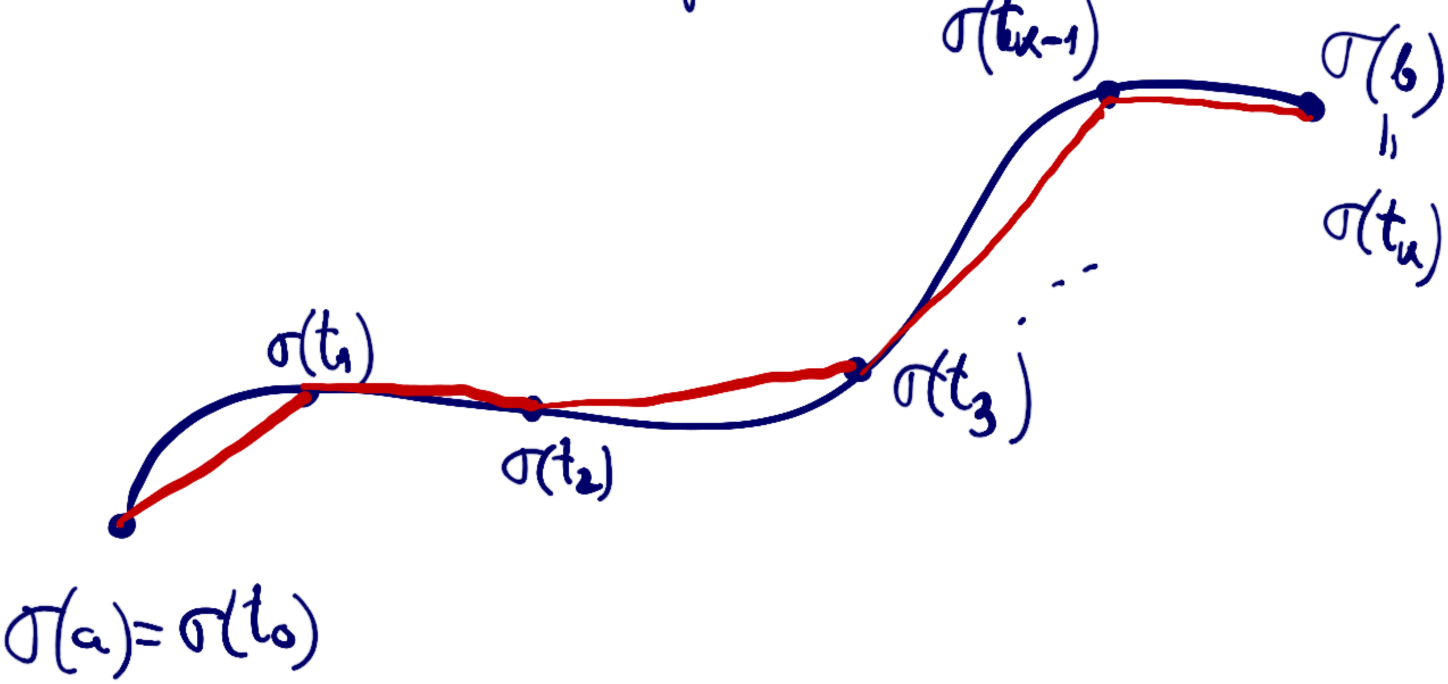
Now take a partition of $[a,b]$

$$a = t_0 < t_1 < t_2 < \cdots < t_K = b$$

$$\Downarrow$$

$$\sigma(a) = \sigma(t_0),\ \sigma(t_1),\ \ldots\ \sigma(t_K) = \sigma(b)$$

We would like to compute the length of C.

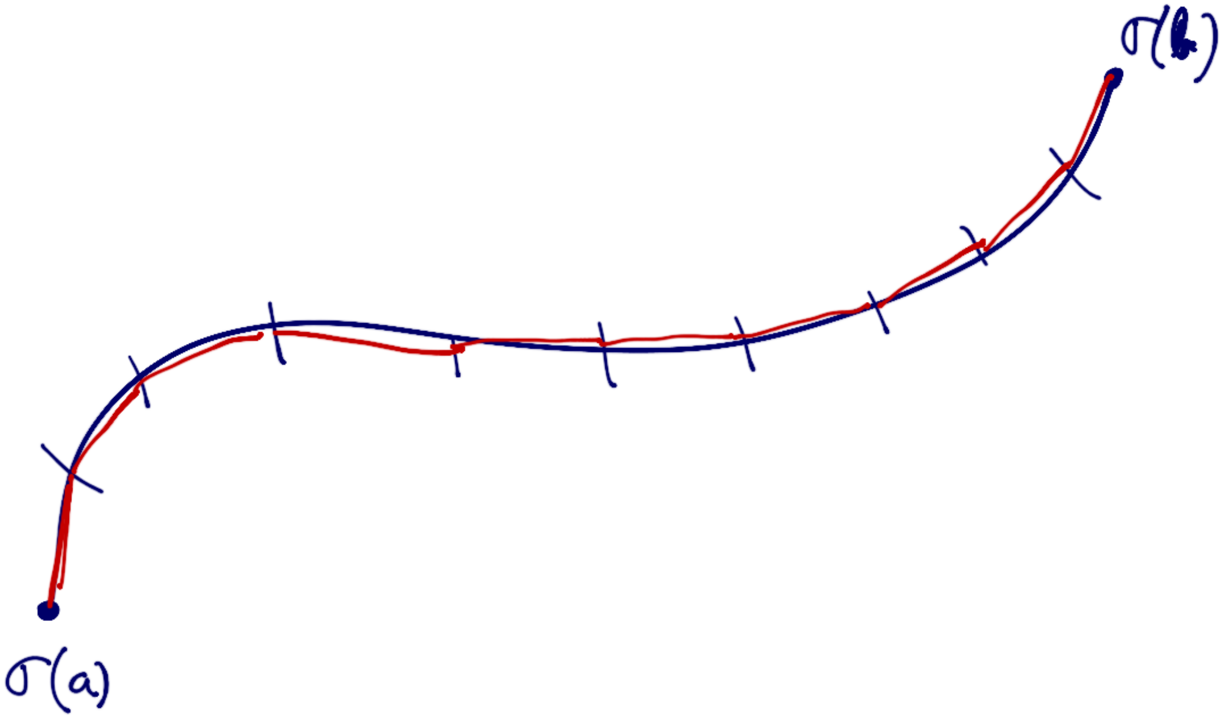


We add those segments.

$$\sum_{i=1}^{k} \|\sigma(t_i)-\sigma(t_{i-1})\| = \overbrace{\sum_{i=1}^{k} \|\sigma'(\xi_i)\|\,(t_i-t_{i-1})}^{\text{Riemann sums for } \|\sigma'(t)\|}$$

using the mean value Th.

$\xi_i \in [t_{i-1}, t_i]$

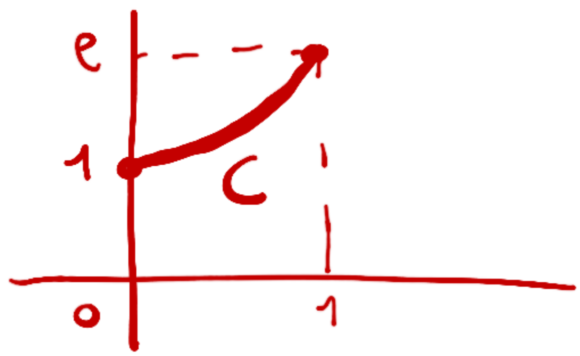


Passing to the limit

$$\lim_{\Delta(t_{i-1},t_i)\to 0} \sum_{i=1}^{k} \|\sigma'(\xi_i)\|(t_i-t_{i-1}) = \int_a^b \|\sigma'(t)\|\,dt = L(\sigma)$$

Example: Find the length of the curve

$f(t)=e^t$ in $[0,1]$



First we parametrise the curve:

$$\left.\begin{array}{l} x=t \\ y=e^t \end{array}\right\} : \sigma \,, \quad \begin{array}{l} \sigma:[0,1]\to\mathbb{R}^2 \\ \quad\; t \to (t,e^t) \end{array}$$

$$\text{length of } C = \int_0^1 \|\sigma'(t)\|\,dt = \boxed{\int_0^1 \sqrt{1+e^{2t}}\,dt}$$

$$\sigma'(t)=(1,e^t)\,, \quad \|\sigma'(t)\|=\sqrt{1+e^{2t}}$$

We choose

$$e^t = x, \quad e^t dt = dx \Rightarrow dt = \frac{dx}{e^t} = \frac{dx}{x}$$

$$\int_0^1 \sqrt{1+e^{2t}}\, dt = \int_1^e \frac{\sqrt{1+x^2}}{x}\, dx = \int_1^e \underline{\sqrt{1+x^2}}\ \underline{\frac{dx}{x}}$$

Now $\boxed{\sqrt{1+x^2} = s} \Rightarrow x^2 = s^2 - 1$

$$\frac{x}{\sqrt{1+x^2}}\, dx = ds \Rightarrow dx = \frac{s}{x}\, ds$$

$$\underbrace{\overbrace{s}^{\sqrt{1+x^2}} \cdot \overbrace{\frac{s}{x^2}\, ds}^{dx/x}}_{=\ \frac{s^2}{x^2}}$$

$$= \int_{\sqrt{2}}^{\sqrt{1+e^2}} \frac{s^2}{s^2-1}\, ds$$

$$= \int_{\sqrt{2}}^{\sqrt{1+e^2}} \left(1 + \frac{1}{s^2-1}\right) ds = \int_{\sqrt{2}}^{\sqrt{1+e^2}} \left[1 - \frac{1}{2}\left(\frac{1}{s+1} - \frac{1}{s-1}\right)\right] ds$$

$$= \sqrt{1+e^2} - \sqrt{2} - \frac{1}{2} \log \frac{s+1}{s-1} \Big]_{\sqrt{2}}^{\sqrt{1+e^2}} = \dots = \dots$$