Example on line integrals for scalar functions:

The aunt of Tom Sawyer asked him to paint both sides of her fence. He thinks she should pay 5 cents every 25 feet\(^2\).

How much can Tom earn with this work under those conditions?

Fence is given by a function \( f(x, y) = 1 + \frac{y}{3} \) along a trajectory \( \gamma(t) = (30 \cos^3 t, 30 \sin^3 t) \).
We have \( p(t) \), \( I = \left[ 0, \frac{\pi}{2} \right] \) just considering the first quadrant.

\[
(0, 30) \quad t = \frac{\pi}{2} \quad p(t) = (30 \cos^3 t, 30 \sin^3 t)
\]

\((30, 0)\) at \( t = 0 \)

We might take a partition of \( [a, b] \)

\( a = t_0, \; \cdots, \; t_n = b \)

\( \Delta t = \frac{b-a}{n} \)

Area for the partition:

\[
\Delta S_j = f(p(t_j)) \cdot \Delta s_j \equiv \text{area of rectangle.}
\]
Adding all of those areas

\[ A = \sum_{j=1}^{n} f(l(t_j)) \cdot \Delta S_j = \sum_{j=1}^{n} \overbrace{f(l(t_j)) \| l'(t) \| dt}^{\text{mean value theorem}} \]

\[ \Delta S_j \to 0 \]

\[ \int_{a}^{b} g(l(t)) \| l'(t) \| dt \]

\[ f(xy) = 1 + \frac{y}{3} \]

\[ l(t) = (30 \cos^3 t, 30 \sin^3 t) \quad t \in [0, \frac{\pi}{2}] \]

\[ f(l(t)) = 1 + 10 \sin^3 t \]

\[ l'(t) = 90 (-\cos^2 t \cdot \sin t, \cos t \sin^2 t) \]

\[ \| l'(t) \| = 90 \sqrt{\cos^4 t \cdot \sin^2 t + \cos^2 t \cdot \sin^4 t} = 90 \cos t \cdot \sin t \]
\[ \int_0^{\pi/2} (1 + 10 \sin^3 t) \, 90 \sin t \cos t \, dt \]

\[ = 90 \int_0^{\pi/2} (\sin t \cos t + 10 \sin^4 t \cos t) \, dt \]

\[ = 90 \left[ \frac{\sin^2 t}{2} + 2 \sin^5 t \right]_0^{\pi/2} = 90 \left( \frac{1}{2} + 2 \right) \]

\[ = \frac{450}{2} = 225 \quad \text{(one side of the fence)} \]

\[ \text{Total area} = 4 \cdot 225 = 900 \quad \text{feet}^2 \]

\[ \text{Tom earns} = \frac{900}{25} \cdot 5 = \frac{900}{5} \]
Path integrals for vector fields

\[ \sigma : [a,b] \to \mathbb{R}^n \text{ trajectory.} \]

\[ F : \mathbb{R}^n \to \mathbb{R}^n \text{ vector field.} \]

We would like to compute the line integral of \( F \) along \( \sigma ([a,b]) = \text{curve} \).

\[ F : \sigma ([a,b]) \to \mathbb{R}^n \]

**Definition**

\[ \int_{\sigma} \left( \frac{\sigma'(t)}{\text{vector}} \cdot \frac{\sigma'(t)}{\text{vector}} \right) \, dt \]

**Notation**

\[ \int_{\sigma} F = \int_{\sigma} F(x_1, \ldots, x_n) \cdot ds = \int_{\sigma} F_1 \, dx_1 + \cdots + F_n \, dx_n \]

\[ ds = \sigma'(t) \, dt \]

\[ F \cdot ds \]
Example: Find the path integral of

\[ \mathbf{F}(x,y,z) = (e^y, e^x, e^2) \]

along \( \mathbf{\sigma}(t) = (0, t, t^2) \), \( t \in [0, \log 2] \)

\[ \oint_{\mathbf{\sigma}} \mathbf{F} = \int_0^{\log 2} \mathbf{F}(\mathbf{\sigma}(t)) \cdot \mathbf{\sigma}'(t) \, dt \]

\[ \mathbf{F}(\mathbf{\sigma}(t)) = (e^t, 1, e^{t^2}) \]

\[ \mathbf{\sigma}'(t) = (0, 1, 2t) \]

\[ \oint_{\mathbf{\sigma}} \mathbf{F} = \int_0^{\log 2} \left( 1 + 2te^{t^2} \right) \, dt = \left. \left( t + e^{t^2} \right) \right|_0^{\log 2} \]

\[ = \log 2 + e^{(\log 2)^2} - 1 \]
Relation between line integrals for scalar fields and vector fields

\[ \int_{\gamma} F \cdot ds = \int_{a}^{b} F(\sigma(t)) \cdot \sigma'(t) \, dt \]

\( \gamma: [a,b] \rightarrow \mathbb{R}^n \) trajectory.

\[ = \int_{a}^{b} \left( F(\sigma(t)) \cdot \frac{\sigma'(t)}{\|\sigma'(t)\|} \right) \cdot \|\sigma'(t)\| \, dt \]

\( \text{scalar product} \)

\( g(\sigma(t)) \) scalar function.

\[ = \int_{a}^{b} g(\sigma(t)) \cdot \|\sigma'(t)\| \, dt \]
Parametrizations

We do not have a unique parametrization.

Example: \( \{(t, t^2), \ t \in [0, 1] \} \)

\[
y = x^2 \quad \text{if} \quad x = t \quad \text{te} \quad [0, 1]
\]

\[
\{(2t, (2t)^2), \ t \in [0, \frac{1}{2}] \}
\]

Both represent the same curve.

What happens with the line integral depending on the parametrization?
Proposition

Two parametrizations of the same curve \( \sigma, \rho \)
\( f: \mathbb{R}^n \rightarrow \mathbb{R} \) scalar field.

Then,
\[
\int_{\sigma} f = \int_{\rho} f
\]

Proposition

Two param. \( \sigma \) and \( \rho \). \( F: \mathbb{R}^n \rightarrow \mathbb{R}^n \) vector field.

a) If \( \sigma \) and \( \rho \) have the same orientation
\[
\int_{\sigma} F = \int_{\rho} F
\]

b) If \( \sigma \) and \( \rho \) have opposite direction
\[
\int_{\sigma} F = - \int_{\rho} F
\]
Example: Circle \( x^2 + y^2 = 1 \)

\[ \gamma : [0, 2\pi] \rightarrow \mathbb{R}^2 \]

\[ t \rightarrow (\cos t, \sin t) = \gamma(t) \]

anticlockwise = positive orientation

\[ \chi : [0, 2\pi] \rightarrow \mathbb{R}^2 \]

\[ t \rightarrow \chi(t) = (\cos(2\pi - t), \sin(2\pi - t)) \]

clockwise = negative orientation.
at \( t = 0 \) \( \Phi(0) = (1, 0) \)

at \( t = \frac{\pi}{2} \) \( \Phi(\frac{\pi}{2}) = (0, -1) \)

at \( t = \pi \) \( \Phi(\pi) = (-1, 0) \)

at \( t = 2\pi \) \( \Phi(2\pi) = (1, 0) \)

\[
\int_{a}^{b} F = \int_{a}^{b} F(\sigma(t)) \cdot \sigma'(t) \, dt
\]

\[
\int_{a}^{b} \int_{0}^{1} F(\sigma(t)) \cdot \sigma'(t) \, dt
\]
Fundamental Theorem of Calculus

\[ \sigma : [a,b] \rightarrow \mathbb{R}^n \text{ trajectory, } \sigma \in C^1 \]

and

\[ f : \sigma([a,b]) \rightarrow \mathbb{R}^n \text{ scalar field along } \sigma \quad f \in C^1 \]

Then,

\[ \int \nabla f \cdot ds = f(\sigma(b)) - f(\sigma(a)) \]

Remark:

Important because \( \nabla f \equiv \text{vector field} \).

so we can define

\[ F = \nabla f \]

If \( \sigma \) is a closed curve

\[ \int_\sigma \nabla f \cdot ds = \int_\sigma F \cdot ds = 0 \]
Example: We would like to compute

\[ \oint_C \mathbf{F} \cdot d\mathbf{s} = \int_C y \, dx + x \, dy \quad \text{with} \quad \sigma(t) = (t^q, \sin \left(\frac{\pi t}{2}\right)) \]

\[ t \in [0, 1] \]

\[ \mathbf{F}(x, y) = (y, x) \]

\[ \mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{vector field.} \]

\[ \oint_C \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\sigma(t)) \cdot \sigma'(t) \, dt = \int_0^1 \left( t^q \sin \left(\frac{\pi t}{2}\right) + \cdots \right) \]

\[ \mathbf{F}(\sigma(t)) = \left( \sin^q \left(\frac{\pi t}{2}\right), t^q \right) \]

\[ \sigma'(t) = \left( q t^{q-1}, q \sin \left(\frac{\pi t}{2}\right) \cos \left(\frac{\pi t}{2}\right) \cdot \frac{\pi}{2} \right) \]

Substituting we get to a very complicated integral.
Looking the vector field.

\[ F(x, y) = \nabla f(x, y) \quad \text{with} \quad f(x, y) = xy \]

\[ F(x, y) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (y, x) \]

Applying FTC.

\[ \int_c F \cdot ds = \int_0^1 \nabla f \cdot ds = f(\sigma(1)) - f(\sigma(0)) \]

\[ \int_0^1 \nabla f(\sigma(t)) \cdot \sigma'(t) \, dt \]

\[ \sigma(0) = (0, 0) \Rightarrow f(\sigma(0)) = f(0, 0) = 0 \]

\[ \sigma(1) = (1, 1) \Rightarrow f(\sigma(1)) = f(1, 1) = 1 \]
In 1D

\[ \int f(x) \, dx = F(x) + C \quad \text{antiderivative} \]

\[ F'(x) = f(x) \]

Barrow's Law

\[ \int_{a}^{b} F'(x) \, dx = F(b) - F(a) \]

equivalent to

\[ \int_{\sigma(a)}^{\sigma(b)} \nabla f \cdot ds = f(\sigma(b)) - f(\sigma(a)) \]
**Definition**

If \( F = \nabla f \), then we say that \( F \) is a **conservative field**

**Corollary** (Consequence)

\( \sigma : [a,b] \rightarrow \mathbb{R}^n \) trajectory of a closed curve, \( \sigma(a) = \sigma(b) \), \( \sigma([a,b]) \in C^1 \)

\( f : \mathbb{R}^n \rightarrow \mathbb{R} \) scalar field, \( f \in C^1 \)

Then \( \int_{\sigma} \nabla f = 0 = f(\sigma(b)) - f(\sigma(a)) \), FTC.
Definition

\[ F : \Omega \to \mathbb{R}^n \] vector field and \( \Omega \subset \mathbb{R}^n \) with \( F \) having an exact differential form

\[ \text{conervative field.} \]

Then there exists a scalar field

\[ f : \mathbb{R}^n \to \mathbb{R} \]

such that \( \nabla f = F \) in \( \Omega \)

\( f \) is called a potential function.

Ex: \( F \) force.

\[ F = -\nabla V, \quad V = \text{potential function}. \]
**Theorem**

D is an open set simply connected

- any closed curve can be deformed continuously inside D
- in $\mathbb{R}^2, \mathbb{R}^3$ this means there are no holes.

$F \in C^1(D)$ vector field.

Then the following is equivalent:

a) $F$ is conservative $\nabla f = F$

b) For any closed curve $\int_F = 0$

c) If we have two different parameterizations $\int_{C_1} F = \int_{C_2} F$ $F$ is conservative.
d) In $\mathbb{R}^2$ if $\mathbf{F} = (P, Q)$

\[
\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \sim \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}
\]

since $\mathbf{F} = \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right) = (F_1, F_2)$

then we have.

\[
\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}
\]

Schwarz TH.

e) In $\mathbb{R}^3$

\[
\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = 0 \quad \text{equivalent to d) in } \mathbb{R}^2
\]
\[
\text{rot} \mathbf{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_1 & F_2 & F_3
\end{vmatrix}
\]

\[
= \mathbf{i} \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) - \mathbf{j} \left( \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) + \mathbf{k} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)
\]

If we are in \( \mathbb{R}^2 \), \( F_3 = 0 \) for

\[
\mathbf{F} = (F_1, F_2, F_3) = (F_1, F_2, 0)
\]

and \( F = F_1(x, y) \) and \( F_2 = F_2(x, y) \)

then

\[
\text{rot} \mathbf{F} = (0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})
\]

\[d\) \Rightarrow \text{rot} \mathbf{F} = 0]
Problem 1 - Set 4.2

\[ F(x, y, z) = (\sin y + z, x \cos y + e^z, x + ye^z) \]

a) Show that \( F \) is conservative.

\[
\text{rot} \boldsymbol{F} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\sin y + z & x \cos y + e^z & x + ye^z
\end{vmatrix}
\]

\[
= \mathbf{i} \left( \frac{\partial}{\partial y} (x + ye^z) - \frac{\partial}{\partial z} (x \cos y + e^z) \right) \\
- \mathbf{j} \left( \frac{\partial}{\partial x} (x + ye^z) - \frac{\partial}{\partial z} (\sin y + z) \right) \\
+ \mathbf{k} \left( \frac{\partial}{\partial x} (x \cos y + e^z) - \frac{\partial}{\partial y} (\sin y + z) \right)
\]

= 0
\[= i (e^z - e^z) - j (1 - 1) + k (\cos y - \cos y)\]

\[= 0 \text{ (vector)}\]

\[\nabla \cdot F \text{ is conservative} \implies \exists \phi \text{ such that } F = \nabla \phi\]

b) Compute \(\phi : \mathbb{R}^3 \to \mathbb{R}\).

\(F = \nabla \phi = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)\)

\[= (\sin y + z, x \cos y + e^z, x + ye^z)\]

so that

\[\frac{\partial \phi}{\partial x} = \sin y + z, \quad \frac{\partial \phi}{\partial y} = x \cos y + e^z\]

\[\frac{\partial \phi}{\partial z} = x + ye^z\]
\[ \frac{\partial \phi}{\partial x} = \sin y + z \implies \phi(x, y, z) = \int (\sin y + z) \, dx \]

\[ \phi(x, y, z) = x \sin y + x^2 + A(y, z) \]

\[ \frac{\partial \phi}{\partial y} = x \cos y + e^z \implies \phi(x, y, z) = \int (x \cos y + e^z) \, dy \]

\[ \phi(x, y, z) = x \sin y + y e^z + B(x, z) \]

\[ \frac{\partial \phi}{\partial z} = x + y e^z \implies \phi(x, y, z) = \int (x + y e^z) \, dz \]

\[ \phi(x, y, z) = x^2 + y e^z + C(x, y) \]

\[ \phi(x, y, z) = x \sin y + x^2 + y e^z + K \quad \text{Potential function} \]