

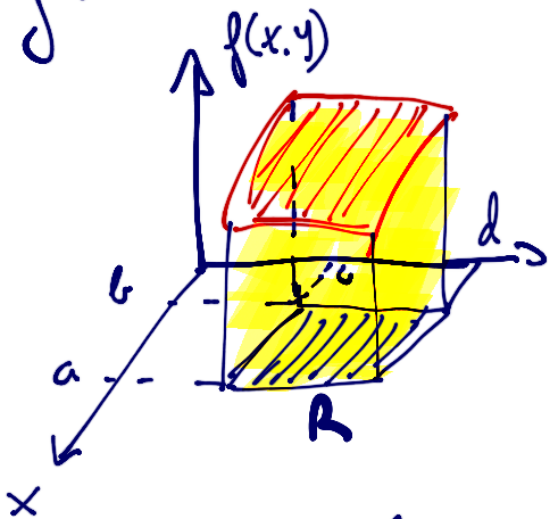
Integration $\mathbb{R}^N$

We analyse integrals in $\mathbb{R}^2$ and $\mathbb{R}^3$

- Double integrals
- Triple integrals.

Double integrals

$f: \mathbb{R}^2 \rightarrow \mathbb{R}$ a scalar function.

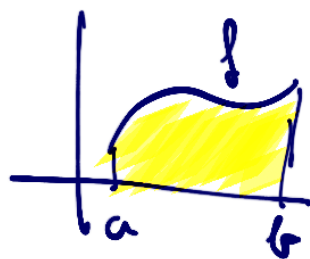


Consider the function f on the region/rectangle.

$$R = [a, b] \times [c, d]$$

Double integral $\equiv$ Volume under the graph.

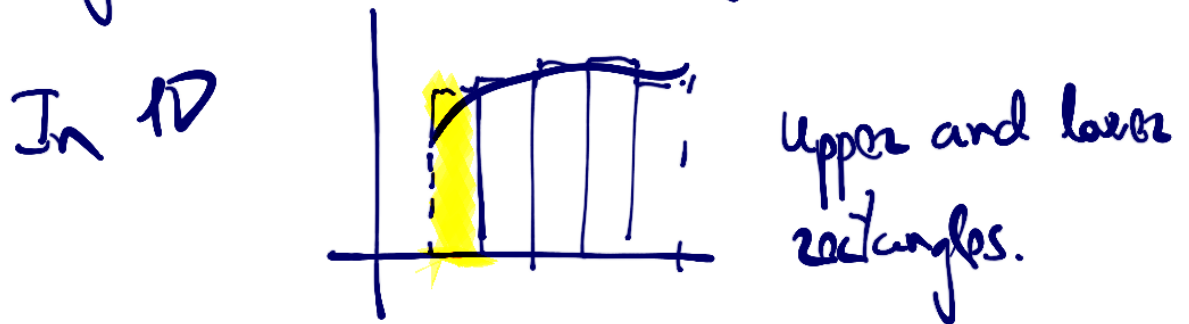
Similar to 1D



$$\int_a^b f \equiv \text{area below the graph.}$$

$$\text{Volume under the graph} = \int_a^b \int_c^d f(x,y) dx dy.$$

We define that Volume using Riemann sums.



In 2D we obtain the integral in a rectangular region $R = [a,b] \times [c,d]$ adding those Riemann sums and applying a limiting process.

Definition

$R \equiv \text{rectangle}$

In $\mathbb{R}^2$, $R = [a_1, b_1] \times [a_2, b_2]$

In $\mathbb{R}^3$, $R = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$

Area or volume or measure of a rectangle.

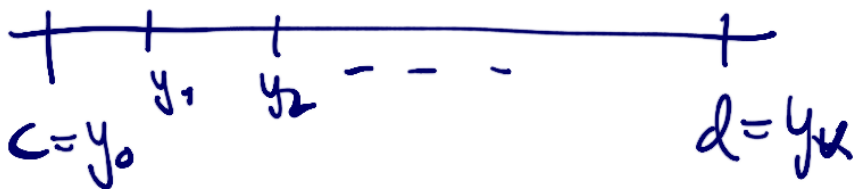
$$|R| = (b_1 - a_1)(b_2 - a_2)(b_3 - a_3) \dots$$

$P \equiv$ partition of R

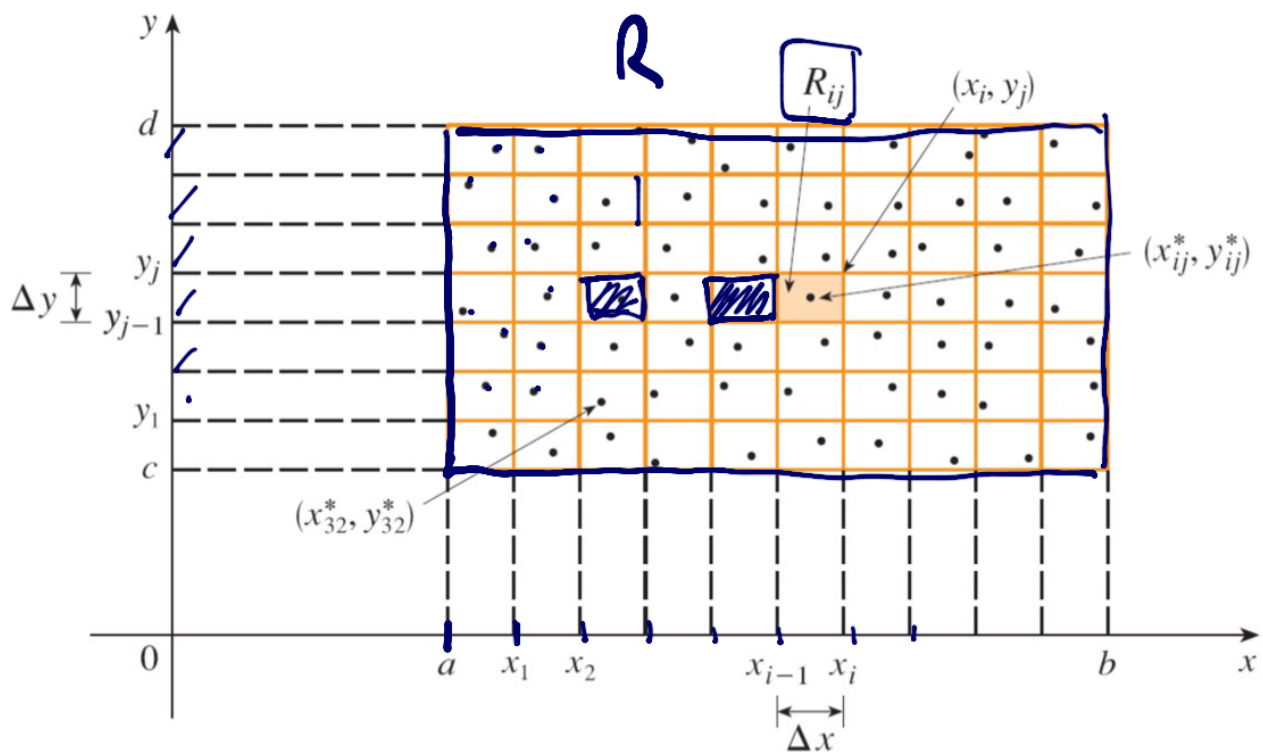
For example in $\mathbb{R}^2$ we have $R = [a, b] \times [c, d]$

$$a = x_0 < x_1 < \dots < x_k = b$$

$$c = y_0 < y_1 < \dots < y_k = d$$



such that $x_{j+1} - x_j = \frac{b-a}{k}$, $y_{j+1} - y_j = \frac{d-c}{k}$



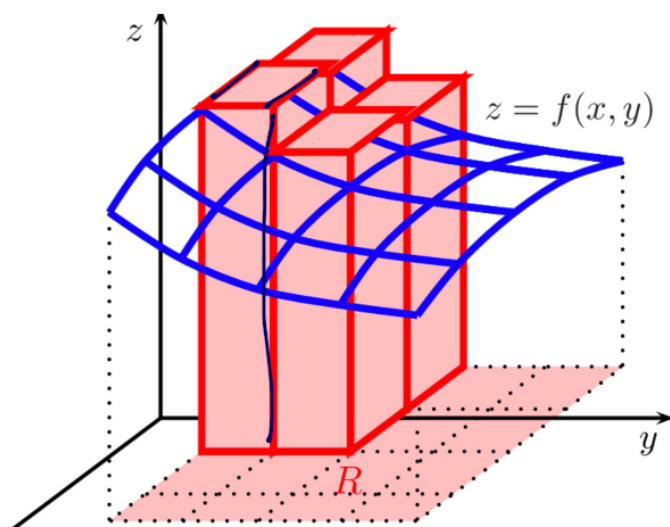
$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$$

Riemann sums for that partition P are:

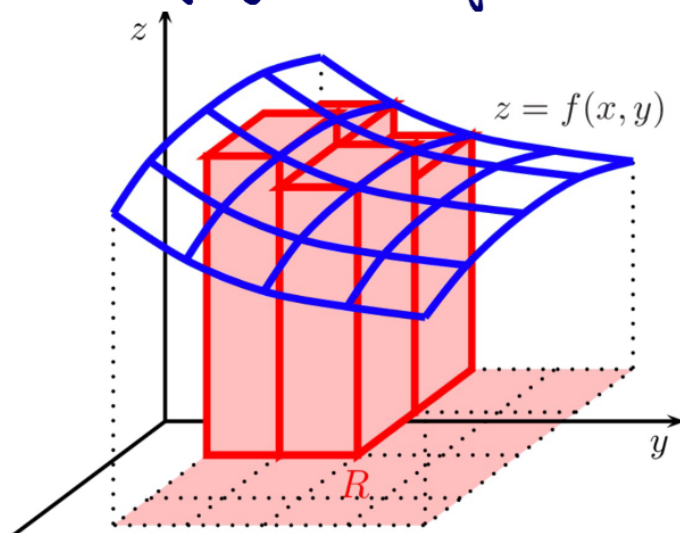
$$\text{Upper Sum of } f = U_f(P) = \sum_{i,j=1}^n M_{ij} |R_{ij}|$$

$$\text{Lower Sum of } f = L_f(P) = \sum_{i,j=1}^n m_{ij} |R_{ij}|$$

where $M_{ij} = \sup \{ f(x,y), (x,y) \in R_{ij} \}$
 biggest of f on R_{ij}
 $m_{ij} = \inf \{ f(x,y), (x,y) \in R_{ij} \}$
 smallest value of f on R_{ij}

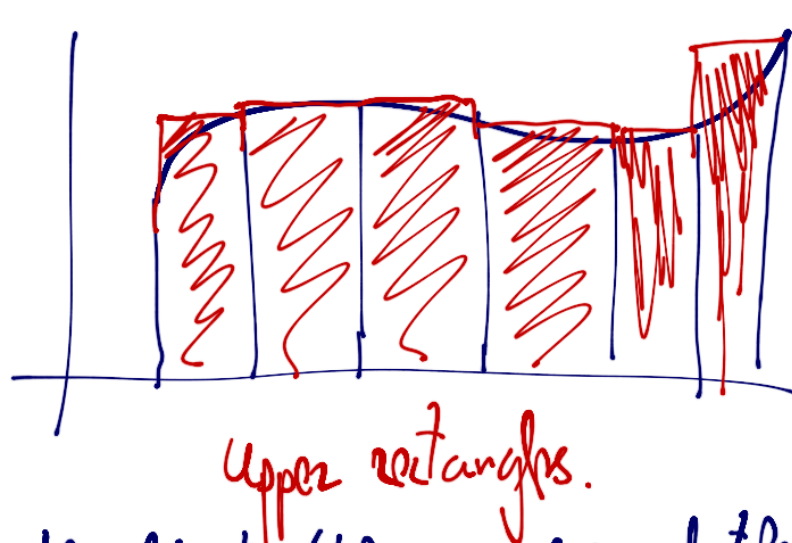
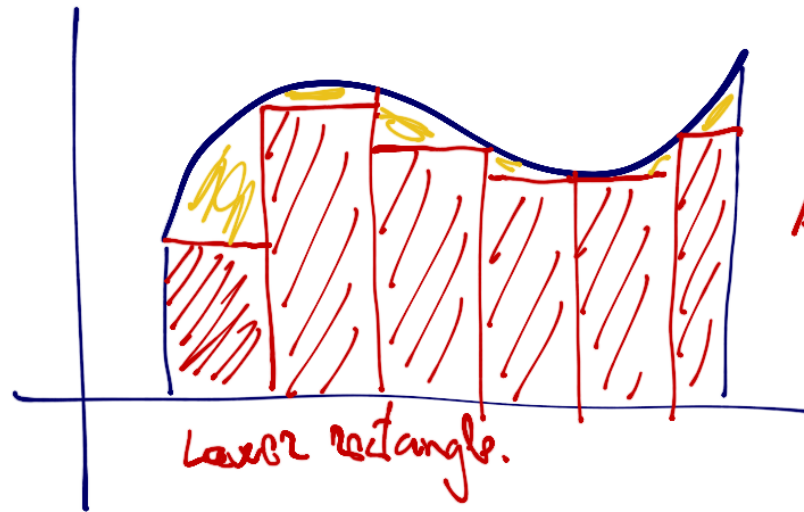


upper sum for the
parallelepipids



lower sum for the
parallelepipids.

$$L(P) \leq \iint_R f \leq U(P)$$



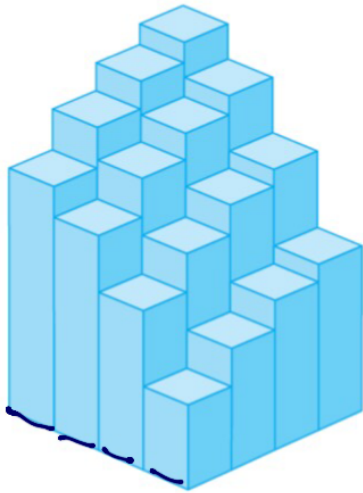
Passing to the limit (the number of the partition)

$$\lim_{k \rightarrow \infty} \sum_{i,j=1}^k M_{ij} |R_{ij}| = \lim_{k \rightarrow \infty} \sum_{i,j=1}^k m_{ij} |R_{ij}| = \iint_R f$$

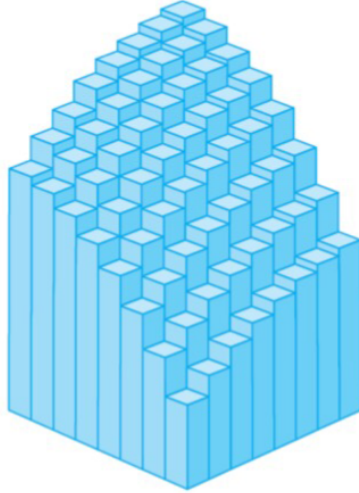
f integrable

$$\iint_R f = \iint_R f(x,y) dx dy = \iint_R f(x,y) dA$$

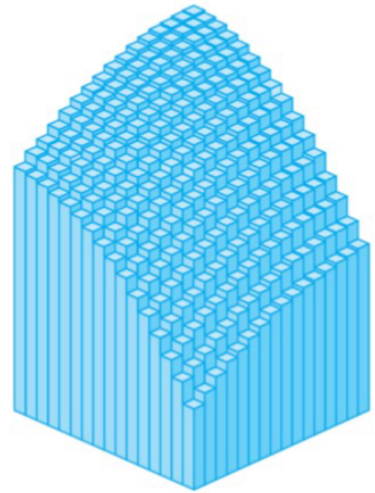
$$z = 16 - x^2 - 2y^2$$



(a) $m = n = 4$, $V \approx 41.5$



(b) $m = n = 8$, $V \approx 44.875$



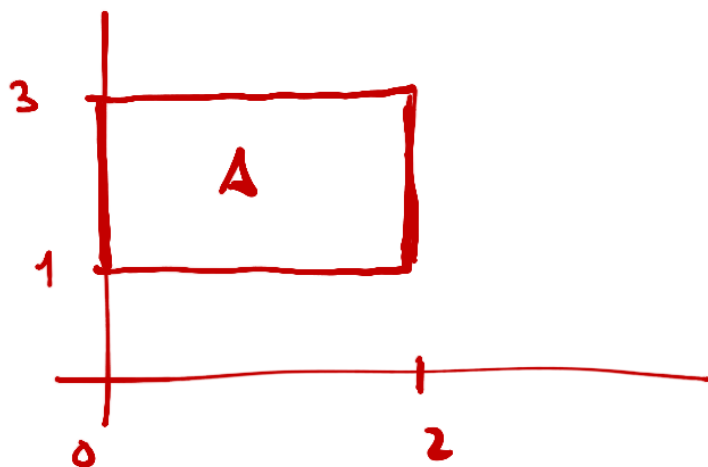
(c) $m = n = 16$, $V \approx 46.46875$

Problem 3 - 3.1

ii) $A = [0, 2] \times [1, 3]$ $f(x, y) = x^2 y$

Proof

$$0 \leq \int_A f \leq 48$$



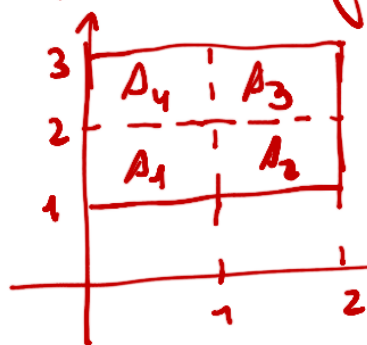
$$0 \leq f(x,y) = x^2 y \leq 4 \cdot 3 = 12$$

Integrating.

$$0 \text{ area of } A \leq \int_A f \leq 12 \cdot \text{Area of } A$$

$$0 \leq \int_A f \leq 12 \cdot (2 \cdot 2) = 12 \cdot 4 = 48$$

iii) Divide A in four equal squares.



$$A_1 = [0, 1] \times [1, 2]$$

$$A_2 = [1, 2] \times [1, 2]$$

$$A_3 = [1, 2] \times [2, 3]$$

$$A_4 = [0, 1] \times [2, 3]$$

$$\int_D f = \int_{A_1} f + \int_{A_2} f + \int_{A_3} f + \int_{A_4} f$$

In $A_1 = [0, 1] \times [1, 2]$, $|A_1| = 1$
 $0 \leq f(x, y) = x^2 y \leq 1 \cdot 2 \leq 2 \Rightarrow 0 \leq \int_{A_1} f \leq 2$

In $A_2 = [1, 2] \times [1, 2]$
 $1 \leq f(x, y) \leq 4 \cdot 2 = 8 \Rightarrow 1 \leq \int_{A_2} f \leq 8$

In $A_3 = [1, 2] \times [2, 3]$
 $2 \leq f(x, y) \leq 4 \cdot 3 = 12 \Rightarrow 2 \leq \int_{A_3} f \leq 12$

In $A_4 = [0, 1] \times [2, 3]$
 $0 \leq f(x, y) \leq 3 \Rightarrow 0 \leq \int_{A_4} f \leq 3$

Then

$$3 \leq \int_A f \leq 2 + 8 + 12 + 3 = 25$$

Triple integrals

$$\int_B f = \iiint_B f(x,y,z) dx dy dz = \iiint_B f(x,y,z) dV$$

Problem 1 - 3.1

$$\int_Q f \quad \text{with} \quad f(x,y) = xy(x+y) \quad Q = [0,1] \times [0,1]$$

$$\int_Q f = \int_0^1 \int_0^1 xy(x+y) dx dy$$

First integral with respect to x

then with respect to y .

$$= \int_0^1 \int_0^1 xy(x+y) dx dy = \int_0^1 \int_0^1 (x^2y + xy^2) dx dy$$

$$= \int_0^1 \left(\frac{x^3}{3} y + \frac{x^2}{2} y^2 \right) \Big|_0^1 dy$$

$$= \int_0^1 \left(\frac{y}{3} + \frac{y^2}{2} \right) dy = \left(\frac{y^2}{6} + \frac{y^3}{6} \right) \Big|_0^1 = \frac{1}{6} + \frac{1}{6}$$

$$= \frac{2}{6} = \frac{1}{3}$$

Fubini's Theorem (Change in the order of integration)

$$R = [a, b] \times [c, d]$$

$$\int_R f = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

Can we compute integrals in non-rectangular regions?

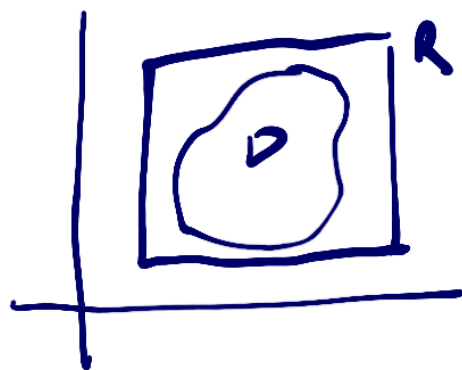
Definition

Let D be a region in $\mathbb{R}^N$ and f a bounded function in D .

Then

$$\int_D f = \int_R f^*$$

such that $D \subset R$



and

$$f^* = \begin{cases} f & \text{in } D \\ 0 & \text{in } R \setminus D \end{cases}$$

Properties f, g bounded in $D \subset \mathbb{R}^n$, $\lambda \in \mathbb{R}$.

Vector
space of
integrable
functions.

i) $f+g$ integrable in D and

$$\int_D (f+g) = \int_D f + \int_D g$$

ii) λf integrable in D and

$$\int_D \lambda f = \lambda \int_D f$$

iii) $|f|$ integrable in D and

$$\left| \int_D f \right| \leq \int_D |f|$$

(Similar to
triangle inequality
 $|a+b| \leq |a| + |b|$)

iv) Cauchy-Schwarz inequality

$$\left(\int_D f \cdot g \right)^2 \leq \left(\int_D f^2 \right) \cdot \left(\int_D g^2 \right)$$

v) $f(x,y) \geq 0 \quad \forall (x,y) \in D$ then

$$\int_D f \geq 0$$

vi) If $f(x,y) \geq g(x,y) \quad \forall (x,y) \in D$ then

$$\int_D f(x,y) dx dy \geq \int_D g(x,y) dx dy$$

We deduce two very important results from those properties.

Theorem

f integrable, continuous in D

If $f \geq 0$ almost everywhere and there exists $x_0 \in D$ where $f(x_0) > 0$

Then,

$$\int_D f > 0$$

Theorem

f integrable. D bounded region and

$$m \leq f(x,y) \leq M$$

Then

$$m(\text{area of } D) \leq \int_D f \leq M(\text{area of } D)$$

Problem 3-3.1

i) Prove that

$$4\pi \leq \int_D (x^2 + y^2 + 1) dx dy \leq 20\pi$$

$D \equiv$ disk of radius 2 and centered at $(0,0)$

$$\left\{ \begin{array}{l} D = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4 \} \\ f(x,y) = x^2 + y^2 + 1 \end{array} \right\}$$

$$\text{Area of } D = 2^2 \cdot \pi = 4\pi$$

$$\text{In } D = \{x^2 + y^2 \leq 4\}$$

$$1 \leq f(x,y) = \underline{x^2 + y^2} + 1 \leq 4 + 1 = 5$$

Then, applying the previous Th.

$$(\text{area of } D) \leq \int_D f \leq 5 \cdot (\text{area of } D)$$

$$4\pi \leq \int_D f \leq 20\pi$$

Theorem - Mean Value Theorem for integrals
 f cont., D bounded, $g \geq 0$ in D . f, g integrable.

a) There exists $c_1 \in D$ such that

$$\int_D f = f(c_1) \cdot (\text{area of } D)$$

b) There exists $c_2 \in D$ such that $\int_D f \cdot g = f(c_2) \int_D g$

Parity

A function is odd in x_j if

$$f(x_1, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_N) = -f(x_1, \dots, x_j, \dots, x_N)$$

A function is even in x_j if

$$f(x_1, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_N) = f(x_1, \dots, x_j, \dots, x_N)$$

Problem 8-3.1

$$\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2) dx dy dz =$$

$$\int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (x^2 + y^2 + z^2) dx dy dz = 2 \int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2) dx dy dz$$

$$= \overbrace{\int_0^1 \int_0^1 \int_0^1 x^2 dx dy dz} + \overbrace{\int_0^1 \int_0^1 \int_0^1 y^2 dy dx dz}$$

$$+ \underbrace{\int_0^1 \int_0^1 \int_0^1 z^2 dz dx dy}$$

$$\int_0^1 \int_0^1 \int_0^1 x^2 dx dy dz = \left(\int_0^1 x^2 dx \right) \cdot \left(\text{area of } [0,1] \right) \cdot \left(\text{area of } [0,1] \right)$$

$$= \int_0^1 x^2 dx$$

$$= 3 \int_0^1 x^2 dx = 3 \frac{x^3}{3} \Big|_0^1 = 1$$

$$ii) \int_0^1 \int_0^1 \int_0^1 (x+y+z)^2 dx dy dz$$

$$= \int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2 + 2xy + 2yz + 2xz) dx dy dz$$

$$= \int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2) dx dy dz$$

$$+ 2 \int_0^1 \int_0^1 \int_0^1 (xy + yz + xz) dx dy dz$$

$$= 1 + 2 \cdot 3 \left(\int_0^1 \int_0^1 \int_0^1 xy \, dx dy dz \right)$$

$$= 1 + 6 \int_0^1 \int_0^1 xy \, dx dy = 1 + 6 \left(\int_0^1 x \, dx \right) \left(\int_0^1 y \, dy \right)$$

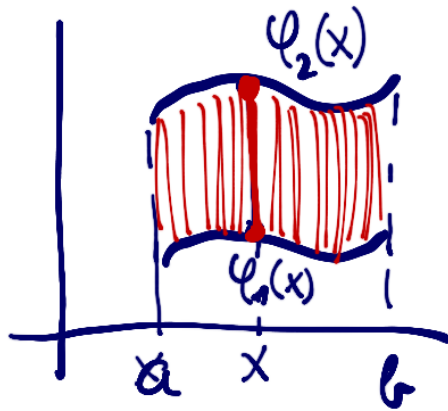
$$= 1 + 6 \left[\frac{x^2}{2} \right]_0^1 \left[\frac{y^2}{2} \right]_0^1 = 1 + \frac{6}{4} = \frac{5}{2}$$

Integration in non-rectangular regions

In 2D we define 3 types of regions in $\mathbb{R}^2$.

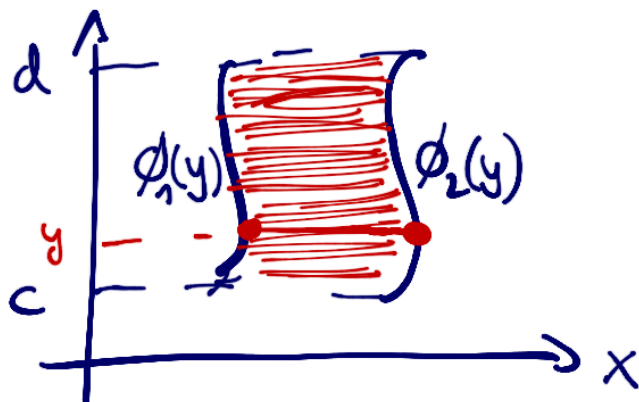
Type 1

$$a \leq x \leq b \text{ and } \varphi_1(x) \leq y \leq \varphi_2(x)$$



$$\iint_D f = \int_a^b \underbrace{\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y) dy}_{F(x)} dx$$

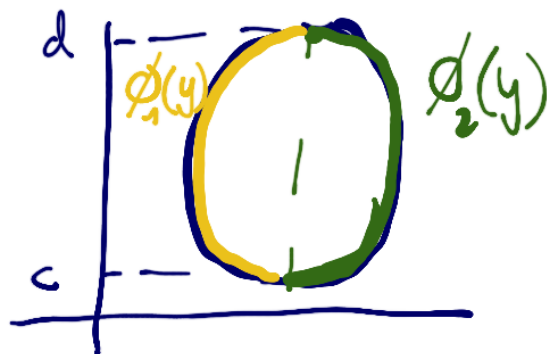
Type 2 $c \leq y \leq d$ and $\phi_1(y) \leq x \leq \phi_2(y)$



$$\iint_D f = \int_c^d \underbrace{\int_{\phi_1(y)}^{\phi_2(y)} f(x,y) dx}_{G(y)} dy$$

Type 3

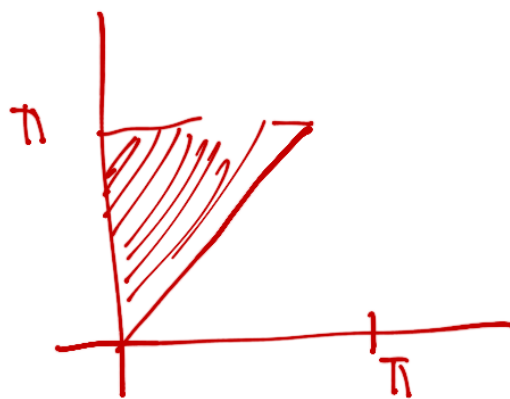
Type 1 and 2 at the same time



Generalizations of Fubini's Theorem for non-rectangular regions

Problem 7-3.1

$$\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y} dy dx$$



There is not primitive for that function.

Integration region

$$D = \{(x, y) \in \mathbb{R}^2, \underbrace{0 \leq x \leq \pi}, \underbrace{x \leq y \leq \pi}\}$$

Type 1

Equivalent to

$$D^* = \{(x, y) \in \mathbb{R}^2, 0 \leq y \leq \pi, \underbrace{0 \leq x \leq y}\}$$

Type 2

$$\begin{aligned} \int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y} dy dx &= \int_0^{\pi} \int_0^y \frac{\sin y}{y} dx dy = \int_0^{\pi} \frac{\sin y}{y} y dy \\ &= \int_0^{\pi} \sin y dy = -\cos y \Big|_0^{\pi} = 2 \end{aligned}$$