Derivatives:  Partial derivatives

If \( f(x,y) \), then

\[
\frac{\partial f(x,y)}{\partial x}, \quad \frac{\partial f(x,y)}{\partial y}
\]

Ratio of change on the direction of the axis.

Directional derivatives:

\[
\nabla f(x,y), \quad \sigma = \text{vector.}
\]

Ratio of change on the direction \( \sigma \).

Existence of all directional derivatives \( \Rightarrow \) differentiability at a tangent plane.
Differentiability

We have \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \) a scalar function and we would like to see if \( f \) is differentiable at a point \( P \).
At a point \((x_0, y_0, f(x_0, y_0))\) \(\in \mathbb{R}^3\) will have a plane going through that point with equation

\[
\frac{z = f(x_0, y_0) + A(x - x_0) + B(y - y_0)}{A, B \in \mathbb{R}\text{ coefficients for the plane.}}
\]

But, if we want to have the tangent plane to the surface \(z = f(x, y)\) at \(P = (x_0, y_0, f(x_0, y_0))\), we need that

\[
\left. \frac{(x-x_0)}{A} \quad \frac{(y-y_0)}{B} \right|_{(x_0, y_0)} = \left. \frac{f(x, y_0)}{A} \quad \frac{f(x_0, y)}{B} \right|_{(x_0, y_0)}
\]

\[
\left[ f(x + h) = f(x) + f'(x_0)(x - x_0) + o(h) \right]
\]

\[
\lim_{h \to 0} \frac{o(h)}{h} = 0
\]
That would be the tangent plane if
\[
\lim_{(x,y) \to (x_0,y_0)} \frac{2(x+y)}{||(x,y)||} = 0 = \lim_{(x,y) \to (x_0,y_0)} \frac{2((x,y)-(x_0,y_0))}{||(x,y)-(x_0,y_0)||}
\]

**Definition - Differentiability**

Let \( A \subset \mathbb{R}^2 \) be a set in \( \mathbb{R}^2 \) such that \((x_0,y_0) \in A \).

Let \( f : A \subset \mathbb{R}^2 \to \mathbb{R} \) be a scalar function.

Then, \( f \) is differentiable at \((x_0,y_0)\) if

1. \( \frac{df(x,y)}{dx}, \frac{df(x,y)}{dy} \) exist,
2. \( \lim_{(x,y) \to (x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \frac{df(x_0,y_0)}{dx}(x-x_0) - \frac{df(x_0,y_0)}{dy}(y-y_0)}{||(x,y)-(x_0,y_0)||} = 0 \)
We might write that limit as

\[
\lim_{(xy) \to (x_0,y_0)} \frac{f(xy) - f(x_0y_0) - Df(x_0y_0) \cdot (x-x_0, y-y_0)}{||(xy) - (x_0,y_0)||} = 0
\]

If that is the case we write the tangent plane to the graph of \( f \) at \( (x_0, y_0) \) as

\[
z = f(x_0y_0) + \frac{\partial f(x_0y_0)}{\partial x} (x-x_0) + \frac{\partial f(x_0y_0)}{\partial y} (y-y_0)
\]

\( A \)

\( B \)

These two particular coefficients define the tangent plane to \( z = f(xy) \) (slopes in the directions of the axis)
Definition

$\mathbb{R}^n \to \mathbb{R}^m, \ x \in A, \ f : A \to \mathbb{R}^m$

\( f \) is differentiable at \( x \in \mathbb{R}^n \) if

a) all partial derivatives exist at \( x_0 \)

b) \( \lim_{x \to x_0} \frac{\|f(x) - f(x_0) - Jf(x_0)(x-x_0)\|}{\|x-x_0\|} = 0 \)

[Jacobian matrix]
Problem 5 of set 1.2

\[ f(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \]

c) Is differentiable at \((0, 0)\)?

We might study the continuity first at \((0, 0)\).

\[ \lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{(x, y) \to (0, 0)} \frac{2xy}{x^2 + y^2} = \lim_{x \to 0} \frac{2x^2}{x^2(1+x^2)} \]

\[ y = 2x \]

\[ = \lim_{x \to 0} \frac{2x}{1+x^2} = \frac{2x}{1+x^2} \quad \text{The limit depends on the direction so it does not exist.} \]

\[ f \text{ is not continuous at } (0, 0) \Rightarrow f \text{ is not diff.} \]
\[
\lim_{(x,y) \to (0,0)} \frac{2xy}{x^2+y^2} = \lim_{r \to 0} \frac{2r^2 \cos \theta \sin \theta}{r^2} = 2 \cos \theta \sin \theta
\]

The limit depends on the direction \( \theta \) so it does not exist.

Existence of all directional derivatives \( \implies \) continuity \( \implies \) differentiability.

ii) Find \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \) at \((0,0)\)

Show that \( \frac{\partial f}{\partial x} \) is not continuous.

\[
\frac{\partial f(0,0)}{\partial x} = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0 = \frac{\partial f(0,0)}{\partial y}
\]
\[ \lim_{\theta \to 0} \frac{f(\theta, 0) - f(0, 0)}{\theta} = \lim_{\theta \to 0} \frac{0 - 0}{\theta} = 0 \]

\[ f(x,y) = \frac{2xy}{x^2 + y^2} \quad \Rightarrow \quad f(0,0) = \frac{2 \cdot 0 \cdot 0}{0^2 + 0} = 0 \]

\[ f(0,0) = 0 \quad \Rightarrow \quad f(0,0) = \frac{2 \cdot 0 \cdot 0}{0^2 + 0} = 0 \]

\[ f(0,0) = 0 \quad \Rightarrow \quad f(0,0) = \frac{2 \cdot 0 \cdot 0}{0^2 + 0} = 0 \]

Partial derivative with respect to \( x \)

\[ f(x,y) = 2y \]

\[ \frac{\partial f(x,y)}{\partial x} = 2y \left( \frac{x^2 + y^2}{x^2 + y^2} \right) - 2xy \frac{2x}{2x} \]

\[ = \frac{-2x^2 y + 2y^3}{(x^2 + y^2)^2} \]

At \((0,0)\)

\[ \frac{\partial f(0,0)}{\partial x} = 0 \]
\[
\frac{\partial f(x,y)}{\partial x} = \begin{cases} 
-\frac{2x^2y + 2y^3}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\
0 & \text{if } (x,y) = (0,0)
\end{cases}
\]

Continuous at \((0,0)\) if
\[
\lim_{(x,y) \to (0,0)} -\frac{2x^2y + 2y^3}{(x^2+y^2)^2} = 0 = \frac{\partial f(0,0)}{\partial x}
\]

Using polar coordinates. \(x = 2\cos \theta\)
\(y = 2\sin \theta\)

\[
\lim_{(x,y) \to (0,0)} -\frac{2x^2y + 2y^3}{(x^2+y^2)^2} = \lim_{r \to 0} -\frac{2r^5 \cos^2 \theta \sin \theta + 2r^3 \sin^3 \theta}{r^4}
\]

\[
= 2 \lim_{r \to 0} \frac{1}{r^2} \left( \sin^3 \theta - \cos^2 \theta \sin \theta \right)
\]

\[
= \infty
\]

\(\text{unbounded.}\)
Problem 10 i)

\[ f(x, y) = x - y + 2 \]  \{ Find the tangent plane. \}

\( (x_0, y_0) = (1, 3) \)

It linear so that tangent plane = \( f \).

\[ z = f(x_0, y_0) + \frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0) \]

\[ f(x_0, y_0) = 1 - 3 + 2 = 0 \]

\[ \frac{\partial f(x_0, y_0)}{\partial x} = 1 \quad \frac{\partial f(x_0, y_0)}{\partial y} = -1 \]

\[ z = (x - 1) - (y - 3) = x - y + 2 \]

\[ 2 = x - y + 2 \] \( \sqrt{\text{Tangent plane.}} \)
But if is away from \( T \) if we are outside of the ball.
Proposition

$A \subset \mathbb{R}^n$, $x_0 \in A$ and $f: A \to \mathbb{R}$ differentiable at $x_0$ and $u \in \mathbb{R}^n \setminus \{0\}$ for a vector.

Then,

$$Df(x_0) = \sum_{i=1}^{N} \frac{\partial f(x_0)}{\partial x_i} \cdot u_i = \langle Df(x_0), u \rangle$$

$u$ is normalised vector

$||u|| = 1$

$\cdot \langle Df(x_0), (\alpha u) \rangle = \alpha \langle Df(x_0), u \rangle$ if $\alpha \neq 1$

We must have

$||u|| = 1$
Example: Set 1.3 problem 3 i)

\[ f(x,y) = x^2 + y^2 \text{ at } (1,1) \text{ along the direction } (1,-1) \]

\[ \nabla f(1,1) = \lim_{(xy) \to (1,1)} \frac{f((1,1)+t(1,-1)) - f(1,1)}{t \| (1,-1) \|} \]

\( f \) is differentiable everywhere (it is a polynomial)

\[ \nabla f(1,1) = \left< \nabla f(1,1), \frac{(1,-1)}{\| (1,-1) \|} \right> \]

\[ \nabla f(x,y) = (2x, 2y) \implies \nabla f(1,1) = (2, 2) \]

\[ \| (1,-1) \| = \sqrt{1+1} = \sqrt{2} \]

\[ \nabla f(1,1) = \left< (2, 2), \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) \right> = \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{2}} = 0 \]
Remark

- $< \nabla f(x_0), \sigma > = \frac{\| \nabla f(x_0) \| \| \sigma \| \cos(\nabla f, \sigma)}{1}$

= $\| \nabla f(x_0) \| \cos(\nabla f, \sigma) \leq \| \nabla f(x_0) \|

We obtain that the directional derivative of $f$ at $x_0$ in the direction of $\sigma$ is maximal in the direction of $\nabla f$.

$\cos(\nabla f, \sigma) = 1 \implies \text{angle}(\nabla f, \sigma) = 0$

or

$\nabla f \parallel \sigma$

Also,

$-\nabla f(x_0)$ maximal decreasing for $f$. 
Problem 5, set 1.3

Temperature of a metal plate

\[ T(x,y) = e^x \cos y + e^y \cos x \]

a) Direction of maximal increasing for \( T \) at \((0,0)\)

\[ \nabla T(x,y) = \left( e^x \cos y - e^y \sin x, -e^x \sin y + e^y \cos x \right) \]

\[ \frac{\partial T}{\partial x} \quad \frac{\partial T}{\partial y} \]

\[ \nabla T(0,0) = \left( 1, 1 \right) \]

b) \( T \) decreasing the fastest

\[ -\nabla T(0,0) = \left( -1, -1 \right) \]
Proposition

\( \mathbb{R}^n, x_0 \in \Omega, f: \Omega \to \mathbb{R} \) differentiable at \( x_0 \)

with \( \nabla f(x_0) \neq 0 \) then

\( \nabla f(x_0) \perp \) to the level curve of \( f \) at \( f(x_0) \)

For example in \( \mathbb{R}^3 \)

\( \nabla f(x, y, z) \neq (0, 0, 0) \)

\((x, y, z) \in \Omega, \Omega \) domain in \( \mathbb{R}^3 \)

Assume a level curve

\( f(x, y, z) = c \in \mathbb{R} \)

\( S_c = \) specific level curve at \( c \).
Since $f$ is diff., the tangent plane will be perpendicular to the level curve.

So that $\nabla f$ is the normal vector to the level curve.
In other words, take two points on the tangent plane.

\[ P^* = (P_1, P_2, P_3) \quad \text{and} \quad P = (x, y, z) \]

with \( n = (n_x, n_y, n_z) \) as the normal vector to the tangent plane.

\[(P - P^*) \cdot n = 0 \implies (x - P_1)n_x + (y - P_2)n_y + (z - P_3)n_z = 0\]

There is only one normal vector! \\

**Definition** \\
Let \( S \) be a surface in \( \mathbb{R}^3 \), then the tangent plane to \( S \) at \( (x_0, y_0, z_0) \in S \) is given by \( \nabla f(x_0, y_0, z_0) \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = 0 \)
Problem 4.6. Set \( f(x, y, z) = 1.2 \)

\( \nabla f(x, y, z) = (2x, 2y, 2z) \)

\( \nabla f(1, 1, 1) = (2, 2, 2) \)

Tangent plane:

\[ \nabla f(1, 1, 1) \cdot \begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix} = 0 \]

\[ 2x-2 + 2y-2 + 2z-2 = 0 \]

\[ x + y + z = 3 \]