__Proposition__ $f: \mathbb{R}^n \to \mathbb{R}$ diff. at x_0

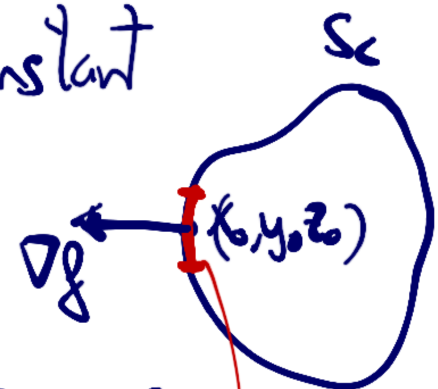
$\nabla f(x_0) \perp$ level curve of f at x_0, $f(x)=f(x_0)$

In $\mathbb{R}^3$, $\nabla f(x,y,z) \neq (0,0,0)$

level curve

$S_c : f(x,y,z) = c$, $c \equiv$ constant

Differentiating at (x_0,y_0,z_0)

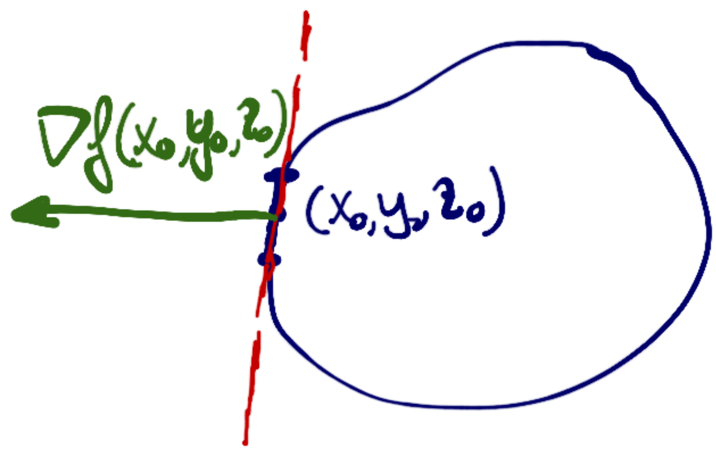


$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0$$

$$\nabla f(x_0,y_0,z_0) \cdot \underbrace{(dx, dy, dz)}_{\text{are close}} = 0$$ to the curve S_c

To construct the tangent plane to the surface $f(x,y,z)$ (two vectors or normal vector)

∇f is orthogonal to the level curve.
and if we are very close to (x_0, y_0, z_0)



we might construct a vector on the tangent plane. taking two points

(x_0, y_0, z_0) and any arbitrary point on the plane (x,y,z)

$$\text{Vector} = (x-x_0,\ y-y_0,\ z-z_0)$$

Therefore, the tangent to the surface.

$$\boxed{\nabla f(x_0,y_0,z_0)\cdot(x-x_0,\ y-y_0,\ z-z_0) = 0}$$

Problem 11

iv) $\boxed{e^{xyz}=1}$ at $(x_0,y_0,z_0)=(1,2,0)$

Tangent plane. $f(x,y,z)=e^{xyz}$

$$\nabla f(x,y,z)=\left(yze^{xyz},\ xze^{xyz},\ xye^{xyz}\right)$$

$$\nabla f(1,2,0)=\left(0,0,2\right)$$

Tangent plane to the surface

$$f(x,y,z)=e^{xyz}$$

at the level curve $f(x,y,z)=1$

will be

$$\nabla f(1,2,0)\cdot\begin{pmatrix}x-1\\y-2\\z\end{pmatrix}=0$$

$$\frac{\partial f(1,2,0)}{\partial x}(x-1)+\frac{\partial f(1,2,0)}{\partial y}(y-2)+\frac{\partial f(1,2,0)}{\partial z}z=0$$

$2z=0 \Rightarrow \boxed{z=0}$

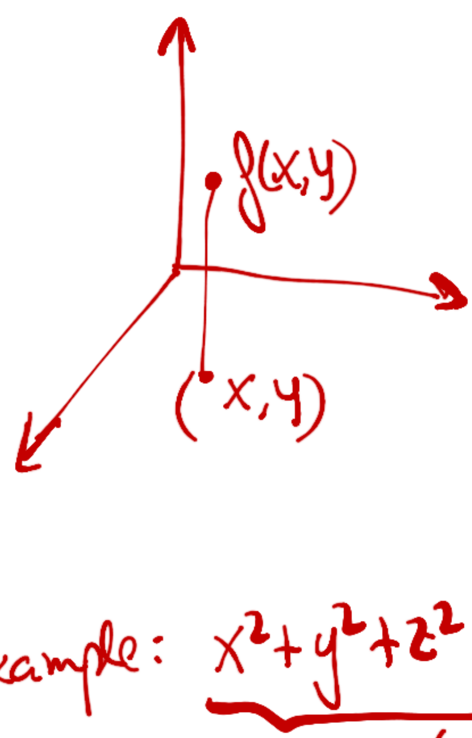
Formula for a tangent plane of a function

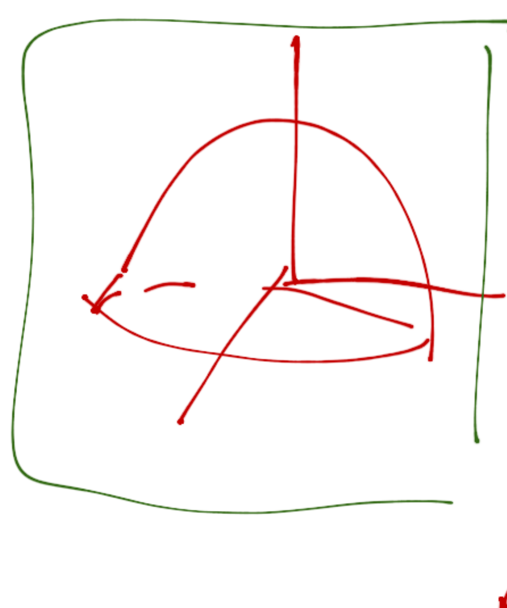
$f(x,y)$

$f:\mathbb{R}^2 \to \mathbb{R}$

$(x,y) \to f(x,y)$

$\sim \underbrace{z=f(x,y)}_{\text{surface.}}$ Formula for the surface.





Example: $\underbrace{x^2+y^2+z^2=3}_{\text{surface (sphere)}}$

but $f(x,y,z)=x^2+y^2+z^2$ is <u>not</u> a function.

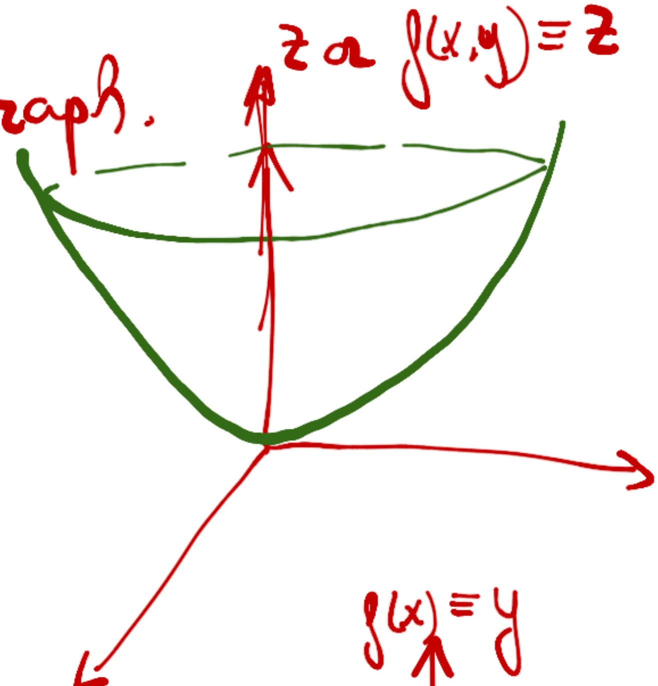
Example: $f(x,y) = x^2+y^2$ function.

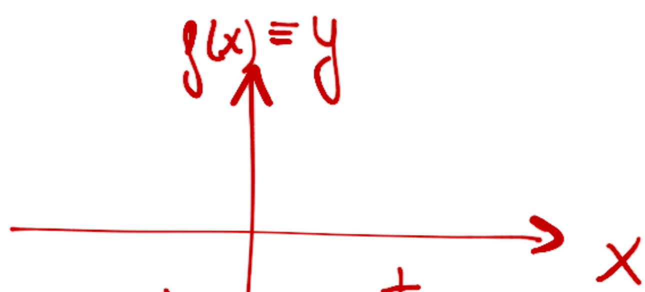
$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x,y) \longrightarrow x^2+y^2 = f(x,y)$$

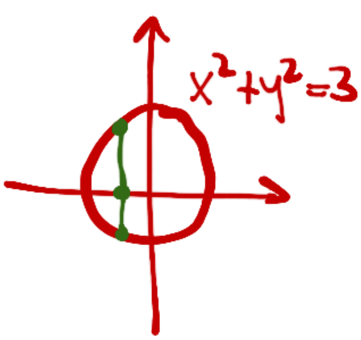
Formula for the surface

$$z = f(x,y) = x^2+y^2$$

Graph.







$x^2+y^2=3$ is not a function.

$$\boxed{z = f(x_0 y_0) + \frac{\partial f(x_0 y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0 y_0)}{\partial y}(y - y_0)}$$

formula for a tangent plane to a function $f(x,y)$ at $(x_0 y_0)$

If $f(x,y) = z$

$f(x,y) - z = g(x,y,z)$ surface.

$\nabla g(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right)$

at $(x_0, y_0, f(x_0,y_0))$ the tangent plane to the surface.

$$\nabla g(x_0, y_0, f(x_0,y_0)) \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - f(x_0 y_0) \end{pmatrix} = 0$$

$$\boxed{\frac{\partial f(x_0 y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0 y_0)}{\partial y}(y - y_0) - z + f(x_0 y_0) = 0}$$

f differentiable, if (f is a function)

$$0 = \lim_{(x,y)\to(x_0,y_0)} \frac{\overbrace{f(x,y) - f(x_0,y_0) - \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) - \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0)}^{\text{Tangent plane.}}}{\|(x,y)-(x_0,y_0)\|}$$

Then, f is very close to the tangent plane at (x_0,y_0)

Theorem

$f: \mathbb{R}^n \to \mathbb{R}^m$, $x_0 \in \mathbb{R}^n$

f differentiable at $x_0 \Rightarrow f$ is continuous at x_0

- However, the existence of all directional derivatives does not guarantee the differentiability.

Theorem

$A \subset \mathbb{R}^n$, $x_0 \in A$, $f: A \to \mathbb{R}$

a) $\exists \dfrac{\partial f}{\partial x_i}$ for any i

b) $\dfrac{\partial f}{\partial x_i}$ continuous at x_0

$\Rightarrow$ f is diff. at x_0.

Example: $f(x,y) = \dfrac{\cos x + e^{xy}}{x^2+y^2}$ f differentiable at any point ?

$$\frac{\partial f(x,y)}{\partial x} = \frac{(x^2+y^2)(-\sin x + y e^{xy}) - (\cos x + e^{xy})2x}{(x^2+y^2)^2}$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{(x^2+y^2)x e^{xy} - (\cos x + e^{xy})2y}{(x^2+y^2)^2}$$

f continuous at $(x,y) \neq (0,0) \Rightarrow f$ diff at $(x,y) \neq (0,0)$

because $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial x}$ are cont at $(x,y) \neq (0,0)$

Proposition

$f, g : A \subset \mathbb{R}^N \to \mathbb{R}^M$, $x_0 \in A$, $\lambda \in \mathbb{R}$.

diff.

Linearity for the derivate.

a) $\lambda f(x)$ diff. at x_0 and $D(\lambda f(x_0)) = \lambda Df(x_0)$

b) $f(x) + g(x)$ diff. at x_0 and

$$D(f(x_0) + g(x_0)) = Df(x_0) + Dg(x_0)$$

c) $D(fg)(x_0) = Df(x_0)\, g(x_0) + f(x_0)\, Dg(x_0)$

d) $$D\left(\frac{f}{g}\right)(x_0) = \frac{g(x_0)\, Df(x_0) - f(x_0)\, Dg(x_0)}{(g(x_0))^2}$$

$g(x_0) \neq 0$

Proposition - Chain rule

Let $f: \mathbb{R}^N \to \mathbb{R}^M$, $g: \mathbb{R}^M \to \mathbb{R}^K$

$$\begin{tikzcd} \mathbb{R}^N \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & \mathbb{R}^M \arrow[r, "g"] & \mathbb{R}^K \end{tikzcd}$$

$$(g \circ f)(x) = g(f(x))$$

$x_0 \in \mathbb{R}^N$, $f(x_0) \in \mathbb{R}^M$

and $g \circ f$ differentiable at x_0

$$D(g \circ f)(x_0) = \underbrace{Dg(f(x_0))} \cdot \underbrace{Df(x_0)}$$

matrices. (Jacobian matrix
or. gradient vector)

Example: Trajectory.

$$\left| \begin{array}{l} g: \mathbb{R} \to \mathbb{R}^3 \\ \quad t \to g(t) = (x(t), y(t), z(t)) \end{array} \right.$$

$$f: \mathbb{R}^3 \to \mathbb{R}$$

and $h(t) = \underbrace{f(g(t))}_{= f(x(t), y(t), z(t))}$, $\frac{dh}{dt}$?

$$\frac{dh(t)}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}$$

$$= Df(g(t)) \cdot \underbrace{g'(t)}_{\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)}$$

Example: $\underline{g(x,y) = (x^2+1, y^2)}$ $\quad g: \mathbb{R}^2 \to \mathbb{R}^2$

$g(1,1) = (2,1)$

$f(u,v) = (u+v, u, v^2)$, $f: \mathbb{R}^2 \to \mathbb{R}^3$

$Dg(x,y) \equiv 2\times2$ Jacobian matrix , $Df(u,v) \equiv 3\times2$ Jacobian matrix

$$Df(u,v) = Jf(u,v) = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2v \end{pmatrix}$$

$$Dg(x,y) = Jg(x,y) = \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}$$

at $(x,y) = (1,1)$, $g(1,1) = \underline{(2,1)}$

$$D(f \circ g)(1,1) = Jf(g(1,1)) \cdot Jg(1,1)$$

$$= Jf(2,1) \cdot Jg(1,1)$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 0 \\ 0 & 4 \end{pmatrix}$$

Chapter 2 - $\boxed{\text{Local extrema}}$

Higher-order derivatives

Partial derivatives of higher-order

$$\boxed{\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)} \qquad \boxed{\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)}$$

Crossing derivatives

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)} , \quad \boxed{\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)}$$

In general, $\dfrac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}$

- We can say that $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is of class C^k if f and all its partial derivatives of order $i = 1, \dots, k$ are continuous

If $\underbrace{f \text{ is of class } C^\infty}_{\text{analytic functions.}}$, f is of class C^k for any $k \in \mathbb{N}$.

(exponentials, cosines, sines)

Example: $f(x,y,z) = e^{xy} + z\cos x$

$$\boxed{\frac{\partial f}{\partial x} = ye^{xy} - z\sin x} \Rightarrow \frac{\partial^2 f}{\partial x^2} = y^2 e^{xy} - z\cos x$$

$$\frac{\partial f}{\partial y} = xe^{xy} \Longrightarrow \frac{\partial^2 f}{\partial y^2} = x^2 e^{xy}$$

$$\frac{\partial f}{\partial z} = \cos x \Longrightarrow \frac{\partial^2 f}{\partial z^2} = 0$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = xy\, e^{xy} + e^{xy}$$

$$\parallel$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = e^{xy} + xy e^{xy}$$

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) = -\sin x$$

$$\parallel$$

$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = -\sin x$$

Crossing derivatives are the same

This is due to regularity of the first derivatives

we see it in the following result.

Schwarz's Theorem

- $f : A \to \mathbb{R}^M$ and $A \subset \mathbb{R}^N$, $\underbrace{f \in C^1}_{\text{regularity condition}}$

Then, if the second order derivatives exist we have that

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \quad \text{for any } 1 \le i, j \le N$$

$$x = (x_1, \dots x_i, \dots x_j, \dots, x_N)$$

- In 2D $f(x, y)$, $f \in C^1$ so $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ cont.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Local extrema

First we define what is a local maximum and minimum.

Definition - local extrema

Let $X \subset \mathbb{R}^n$ be a set in $\mathbb{R}^n$ such that $x_0 \in X$ and $f: X \to \mathbb{R}$.

We say that f reaches its local maximum at x_0 in X if there exists a neighbourhood of x_0, Ω such that

$$f(x) \leq f(x_0) \quad \text{for any } x \in \Omega \setminus \{x_0\}$$

Similarly for the local minimum at x_0 if

$$f(x) \geq f(x_0) \quad \text{for any } x \in \Omega \setminus \{x_0\}$$

Definition

$\Omega \subset \mathbb{R}^N$, $x_0 \in \Omega$, $f : \Omega \to \mathbb{R}$.

We say that $x_0 \in \Omega$ is a stationary point or critical point for f if

$$\boxed{\frac{\partial f(x_0)}{\partial x_i} = 0} \quad \text{for any } 1 \le i \le N$$

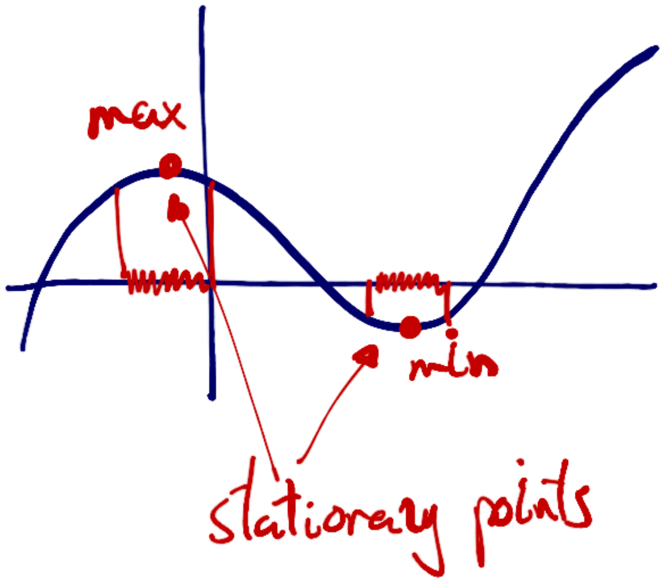
In other words

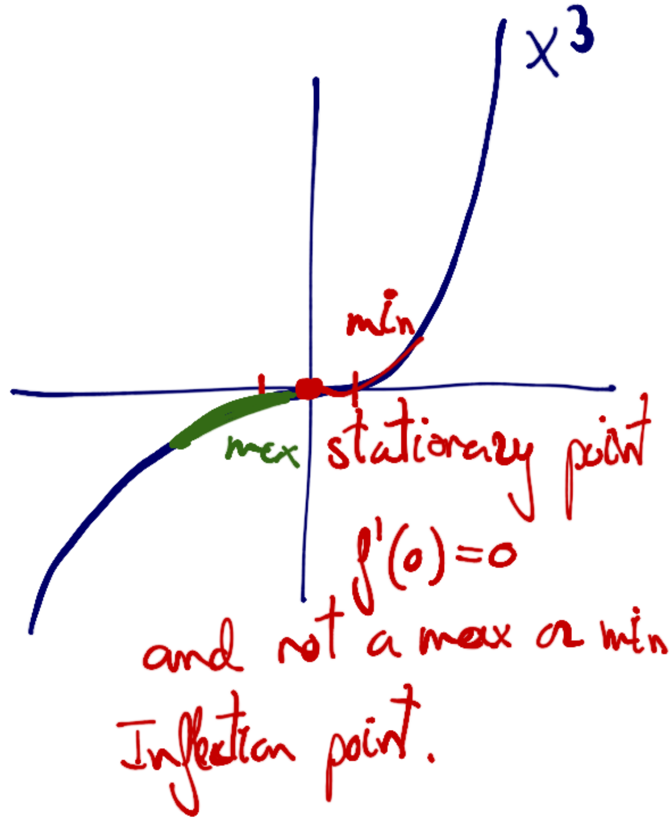
$$\boxed{\nabla f(x_0) = 0}$$

Remark

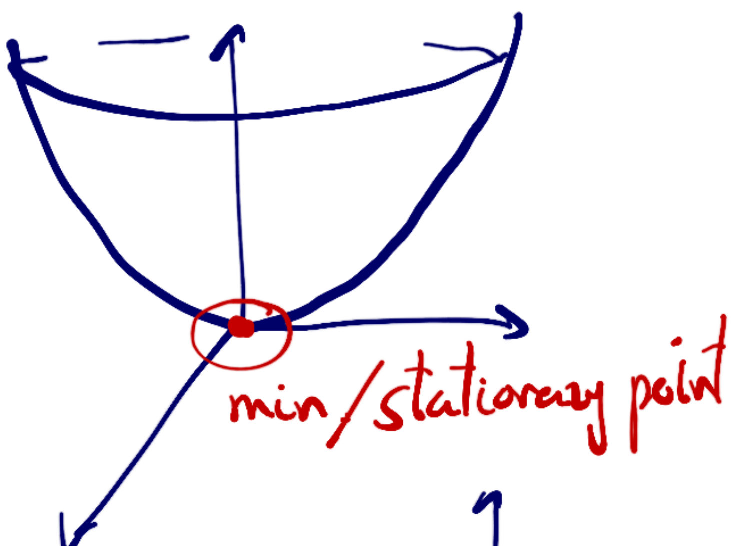
A critical or stationary point which are not local extrema. We call them saddle points

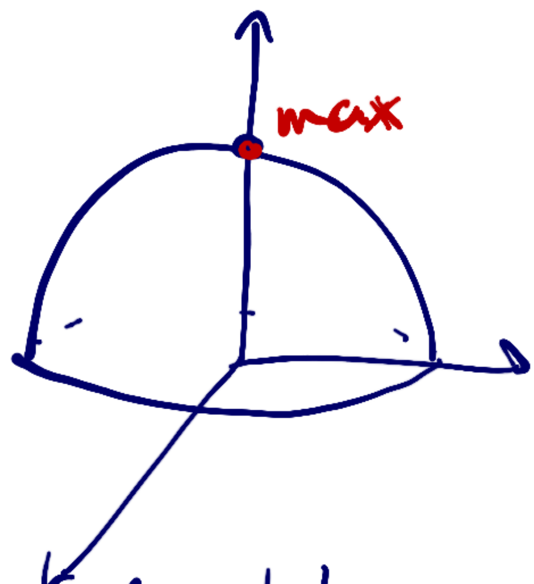
In one variable:

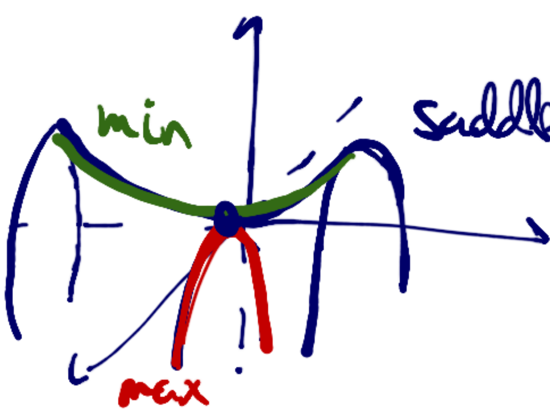




In several dimensions:







saddle point $\equiv$ stationary point.