Ordering

The set of real numbers is equiped with an order relation, denoted by $<$, such that for any given two real numbers a and b, there are three mutually exclusive posibilities:

(i) $a < b$ (a is less than b) $\boxed{b>a}$

(ii) $a = b$ (a equals b)

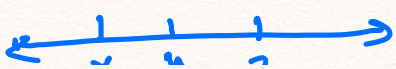
(iii) $b < a$ (b is less than a)

We define

(iv) $a \leq b$ (a is less than or equal to b)

(v) $a \geq b$ (a is greater than or equal to b).

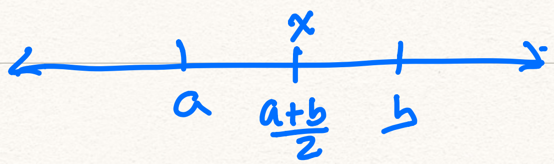
Together, the above relations are called inequalities.

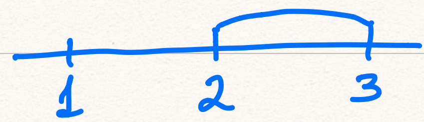


For every $x, y, z \in \mathbb{R}$, the following properties hold:

Transitivity:	If $x < y$ and $y < z$, then $x < z$.
Compatibility with addition:	If $x < y$, then $x + z < y + z$.
Multiplication by a positive factor:	If $x < y$ and $0 < z$, then $x\,z < y\,z$.
Multiplication by a negative factor:	If $x < y$ and $z < 0$, then $x\,z > y\,z$.

__Example:__ Suppose that $\boxed{a<b}$. There exists a real number x satisfying $a<x<b$.





We will prove that $x = \frac{a+b}{2}$ satisfies $a<x<b$.

$$\boxed{a < b} \underset{\substack{\uparrow \\ \text{add } a \\ \text{to both} \\ \text{sides}}}{\Rightarrow} 2a < a+b \underset{\substack{\uparrow \\ \text{divide} \\ \text{by } 2}}{\Rightarrow} a < \frac{a+b}{2} \qquad (1)$$

$$a < b \Rightarrow a+b < 2b \Rightarrow \frac{a+b}{2} < b \qquad (2)$$

Putting (1) and (2) together we get $a < \frac{a+b}{2} < b$

Example: If $b>0$ and $B>0$ and

$$\frac{a}{b} < \frac{A}{B} \xrightarrow[\text{both sides by } bB]{\text{mult.}} \frac{a}{\cancel{b}} \cdot B\cancel{b} < \frac{A}{\cancel{B}} \cdot b\cancel{B} \Rightarrow aB < Ab$$

then $aB < bA$. Deduce that

$$\frac{a}{b} < \frac{a+A}{b+B} < \frac{A}{B}.$$

$$\cancel{\begin{gathered} a(b+B) < (a+A)b \\ a\cancel{b} + aB < a\cancel{b} + Ab \\ aB < Ab \end{gathered}}$$

- $\dfrac{a}{b} < \dfrac{a+A}{b+B}$:

Adding ab to both sides of $aB < Ab$ we get

$$ab + aB < Ab + ab \Rightarrow a(b+B) < (A+a)b \Rightarrow$$

$b+B>0$
$b>0$

$$\frac{a\cancel{(b+B)}}{b\cancel{(b+B)}} < \frac{(A+a)\cancel{b}}{\cancel{b}(b+B)} \Rightarrow \frac{a}{b} < \frac{A+a}{b+B}$$

Complete it on your own!!

$x+y$ $\qquad$ $(x+y)$

Example: $\underbrace{\text{If } a>0,}_{\text{Always true}}$ $\underbrace{\text{then } a^{-1}>0.}_{\text{Prove this is true.}}$

Assume that $a>0$ and $a^{-1} \le 0$.

$\underset{\text{case II}}{a^{-1}<0}$ or $\underset{\text{Case I}}{\cancel{a^{-1}=0}}$.

case I:

Suppose that $a^{-1}=0$. We know that $a \cdot a^{-1}=1$.

But $a^{-1}=0$ so $a \cdot a^{-1}=0 \Rightarrow 0=1$ Contradiction!

So $a^{-1} \neq 0$.

$\left.\begin{array}{r} \to \text{state 1} \\ \text{false} \to \text{state2} \end{array}\right\}$ contradiction

case II:

Now suppose that $a^{-1}<0$. Multiplying both sides by $a>0$, we get $a \cdot a^{-1} < a \cdot 0$ or

$$1 = a \cdot a^{-1} < a \cdot 0 = 0 \Rightarrow 1<0.$$

So $a^{-1}>0$.

Example: If x and y are positive, then $x<y$ if and only if $x^2<y^2$.

- If $x<y$ then $x^2<y^2$ $\Longrightarrow$
- If $x^2<y^2$ then $x<y$ $\Longleftarrow$

- If $x<y$ then $x^2<y^2$

Multiply both sides of $x<y$ times x we get $x^2<xy$.

Multiply both sides of $x<y$ times y we get $xy<y^2$

By transitivity $x^2<y^2$.

$x<y$ $\quad$ $x+y>0$

$x-y<0$

$(x+y)(x-y)<0$

$x^2-y^2<0$

$x^2<y^2$

- If $x^2<y^2$ then $x<y$

$x^2-y^2<0$

$(x+y)(x-y)<0$ (I)

Since $x+y>0$, by the previous

example, $(x+y)^{-1} > 0$. Multiplying both sides of (1) by $(x+y)^{-1}$ we get

$$x-y<0 \xRightarrow[\text{add } y \text{ to both sides}]{} x<y.$$

$$\cancel{(x+y)}^{-1}\,\cancel{(x+y)}\,(x-y) < 0\,(x+y)^{-1} = 0$$

$$x-y<0$$

$$a = x+y > 0$$

$$a^{-1} = (x+y)^{-1} > 0$$