The **image** or **range** of a function is the set $f(A) \equiv \{f(x) : x \in A\}$.

Likewise, we call *image of the set* $C \subset A$ to the set $f(C) \equiv \{f(x) : x \in C\}$.

We call *inverse image of a set* $B \subset \mathbb{R}$ to the set $f^{-1}(B) \equiv \{x \in A : f(x) \in B\}$. Note that $f^{-1}(B) \subset A$.

The **graph** of a function $f(x)$ is the subset of $\mathbb{R}^2$ defined by the points $\{(x, f(x)) : x \in A\}$. Plotting this set is how we represent functions.

A function is **injective,** or **one-to-one,** if for every pair of number $x_1 \neq x_2$ we have $f(x_1) \neq f(x_2)$. If a function is injective, the equation $y = f(x)$ has either no solution or a unique solution.

A function is **surjective,** or **onto,** if $f(A) = \mathbb{R}$. If a function is surjective, the equation $y = f(x)$ always has at least one solution.

A function is **bijective** if it is both injective and surjective. If a function is bijective, the equation $y = f(x)$ always has one, and only one, solution for each $y \in \mathbb{R}$.

A function is **periodic** if there exists $c > 0$ such that $f(x+c) = f(x)$. The smallest such c is referred to as the *period* of the function.

A function is **even** if $f(-x) = f(x)$, and **odd** if $f(-x) = -f(x)$.

A function is **bounded** if there exists $M > 0$ such that $|f(x)| \leqslant M$ for all x in its domain.

A function is **monotonically increasing** if for every x, y in its domain such that $x < y$ it satisfies $f(x) \leqslant f(y)$, and is **monotonically decreasing** if $f(x) \geqslant f(y)$. We say it is **monotonic strictly increasing/decreasing** if inequalities are strict. (Note that a constant is both monotonically increasing and decreasing, but not strictly.)

■ Example 2.2

(a) The domain of $f(x) = x^2$ is $\mathbb{R}$ and its image is $f(\mathbb{R}) = [0, \infty)$. This function is not injective because x and $-x$ have the same square. It is not surjective either because $f(\mathbb{R}) \neq \mathbb{R}$. The inverse image of the interval $[4,9]$ is $f^{-1}([4,9]) = [-3,-2] \cup [2,3]$.

(b) The domain of $f(x) = \log x$ is $(0, \infty)$ and its image is $\mathbb{R}$. It is injective because two different numbers have different logarithms. It is also surjective because any number y is always the logarithm of a number, namely e^y. So it is bijective.

(c) $F(x) = e^x - e^{-x}$ is an odd function because $f(-x) = e^{-x} - e^x = -f(x)$.

(d) $f(x) = \cos x$ is even because $\cos(-x) = \cos x$.

(e) $f(x) = \sin^2 x$ is periodic of period π because $\sin^2(x+\pi) = \sin^2 x$.

■

2.2 Elementary functions

There is a wide range of elementary functions that we will work with. They include polynomials, rational functions, trigonometric functions, the exponential and the logarithm.

2.2.1 Polynomials

These are functions of the form

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \tag{2.2}$$

where $a_k \in \mathbb{R}$ for all $k = 0, 1, \ldots, n$. The largest power, n, is called the *degree* of the polynomial. Constants are polynomials of degree 0. Given the operations that define them, the domain of any polynomial is $\mathbb{R}$.

2.2.2 Rational functions

They are defined as quotients of two polynomials, namely

$$f(x) = \frac{P_n(x)}{Q_m(x)}. \tag{2.3}$$

The domain of both polynomials is $\mathbb{R}$, but $Q_m(x)$ may be zero at some points, where the quotient will thus not be defined. Hence the domain of $f(x)$ is $\{x \in \mathbb{R} : Q_m(x) \neq 0\}$.

2.2.3 Trigonometric functions

The two basic trigonometric functions are the sine ($\sin x$) and the cosine ($\cos x$). In terms of them we can define also the tangent and cotangent:

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}. \tag{2.4}$$

The geometric definition of these functions, based on the unit circle, is described in Figure 2.1.

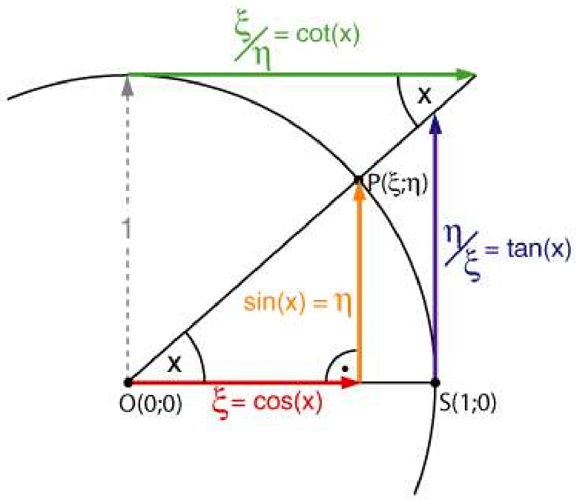

Figure 2.1: Geometric definition of $\sin x$, $\cos x$, $\tan x$, and $\cot x$.

There are two more trigonometric functions, although less common than the previous one, namely the secant ($\sec x$) and the cosecant ($\csc x$):

$$\sec x = \frac{1}{\cos x}, \qquad \csc x = \frac{1}{\sin x}. \tag{2.5}$$

The graphs of $\sin x$ and $\cos x$ are plotted in Figure 2.2. Those of $\tan x$ and $\cot x$ in Figure 2.3.

Trigonometric identities	
$\cos^2 x + \sin^2 x = 1$	$1 + \cot^2 x = \csc^2 x$
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$	$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$
$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$	$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$
$\cos 2x = \cos^2 x - \sin^2 x$	$\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$
$\sin 2x = 2\sin x \cos x$	$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$
$1 + \tan^2 x = \sec^2 x$	$\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$

Table 2.1: Some important trigonometric identities.

Given their geometric definitions, all these functions are related by geometric identities. The main one are listed in Table 2.1.

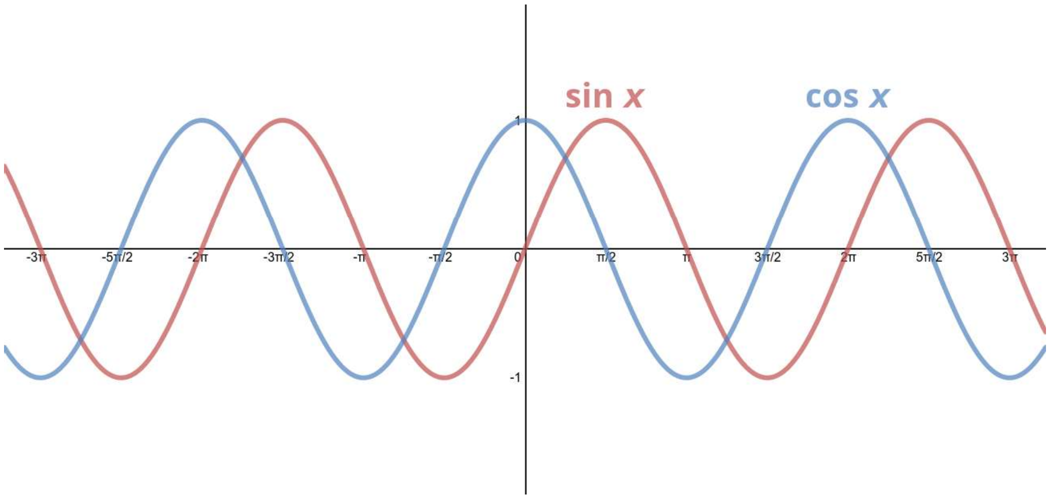

Figure 2.2: Plot of $\sin x$ and $\cos x$.

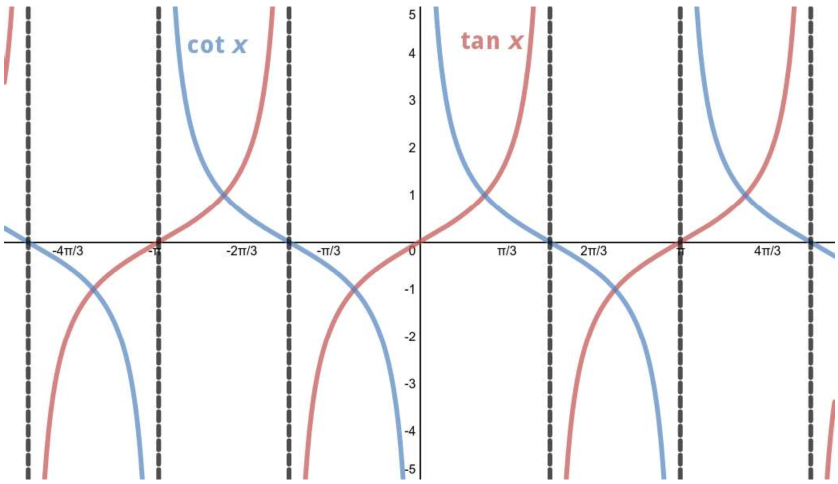

Figure 2.3: Plot of $\tan x$ and $\cot x$.

2.2.4 Exponential

This is the function defined as $f(x) = e^x$. The constant e appearing in this definition is the irrational number introduced by Euler

$$e = 2.71828182845904523536028747135266249775724709369995957496\ldots$$

We will see a proper definition of this constant later on. Apart from that, the definition of the exponential involves raising a real number to a real power. This requires some clarifications.

Integer powers of real numbers are easily defined through the concept of repeated product. Thus $e^3 = e \cdot e \cdot e$. With this definition, for any $n, m \in \mathbb{N}$ it is straightforward that

$$e^{n+m} = e^n e^m, \tag{2.6}$$

from which it follows

$$(e^m)^n = \underbrace{e^m \cdot e^m \cdots e^m}_{n \text{ times}} = e^{m+m+\cdots+m} = e^{nm}. \tag{2.7}$$

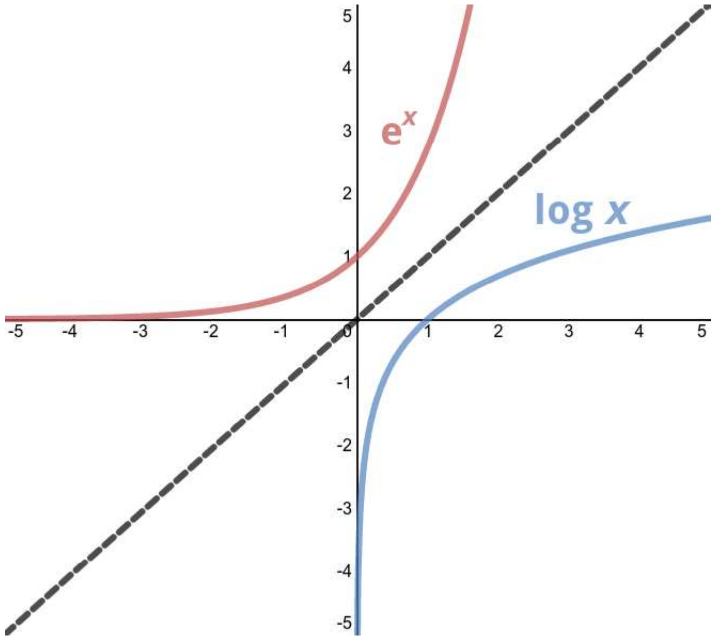

Figure 2.4: Plot of e^x and $\log x$.

We will take these formulas as a basic definition. Extending them will provide meaning to powers other than natural numbers. For instance, applying (2.6),

$$e^{n-m}e^m = e^{n-m+m} = e^n \quad \Rightarrow \quad e^{n-m} = \frac{e^n}{e^m}.$$

But extending (2.6) means assuming $e^{n-m} = e^n e^{-m}$. Cancelling a factor e^n in both sides leads to

$$e^{-m} = \frac{1}{e^m},$$

which provides a meaning to negative powers. And from this definition it follows

$$e^0 = e^{n-n} = \frac{e^n}{e^n} = 1.$$

As for fractional powers, equation (2.7) implies

$$(e^{1/n})^n = e^{n/n} = e \quad \Rightarrow \quad e^{1/n} = \sqrt[n]{e}.$$

Thus, $e^{m/n} = \sqrt[n]{e^m}$. This extension of the basic multiplicative rule provides a definition of the exponential valid for all rational powers. It only remains to define it for irrational powers. But irrational numbers can be approximated as much as we like by rational numbers. If fact, as we have seen, irrational numbers can be bracketed by sequences of rational approximants; i.e., if x is an irrational number, we can find two sequences of rational numbers such that

$$p_1 < p_2 < p_3 < \cdots < p_n < \cdots < x < \cdots < q_n < \cdots < q_3 < q_2 < q_1.$$

Thus we can define e^x as the number bracketed by

$$e^{p_1} < e^{p_2} < e^{p_3} < \cdots < e^{p_n} < \cdots < e^x < \cdots < e^{q_n} < \cdots < e^{p_3} < e^{p_2} < e^{p_1}.$$

Using this definition we can summarise the properties of the exponential as follows:

1. Its domain is $\mathbb{R}$.
2. $e^x > 0$ for all $x \in \mathbb{R}$.
3. It is monotonic strictly increasing —hence injective.
4. $e^0 = 1$.
5. $(e^x)^a = e^{ax}$ for any $a \in \mathbb{R}$.
6. $e^{x+y} = e^x e^y$.
7. $e^{-x} = 1/e^x$.

A plot of the exponential function is shown in Figure 2.4.

Some important functions defined in terms of exponentials define what is known as the *hyperbolic trigonometry*. The main one are the hyperbolic cosine ($\cosh x$) and sine ($\sinh x$), defined as

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}. \tag{2.8}$$

Their plots are shown in Figure 2.5. It can be seen that $\cosh x$ is even whereas $\sinh x$ is odd.

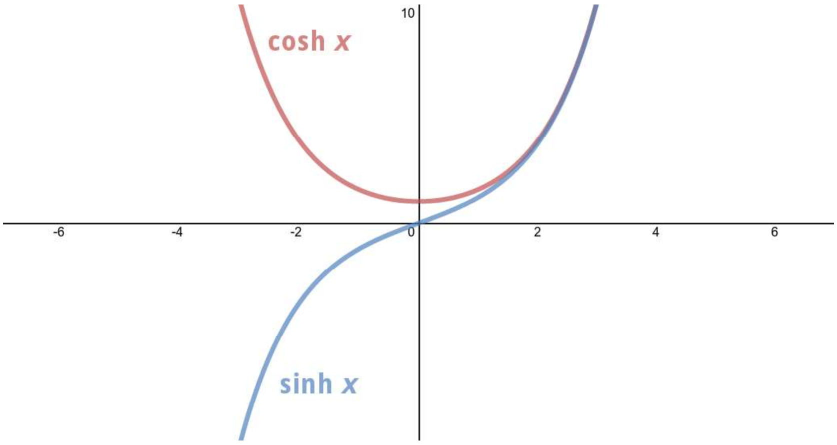

Figure 2.5: Plot of $\cosh x$ and $\sinh x$.

Hyperbolic tangent ($\tanh x$) and cotangent ($\coth x$) can also be defined (see Figure 2.6 for their plots):

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{1}{\tanh x}, \tag{2.9}$$

and similarly $\operatorname{sech} x = 1/\cosh x$ and $\operatorname{csch} x = 1/\sinh x$.

There is a list of identities relating these functions similar to that of the ordinary trigonometry, as illustrated in Table 2.2.

2.2.5 Logarithm

This is the inverse of the exponential. If $y = \log x$ it means that $x = e^y$. Its plot can be seen in Figure 2.4 to mirror that of the exponential with respect to the line $y = x$.

> **R** Along these notes, whenever we write $x = \log y$ we mean that x is the solution of the equation $e^x = y$, in other words, log of a number is the exponent to which we need to rise e in order to obtain that number. In particular $\log 1 = 0$ and $\log e = 1$.

The main properties of the logarithm (derived from those of the exponential) are the following:

1. Its domain is $(0, \infty)$.

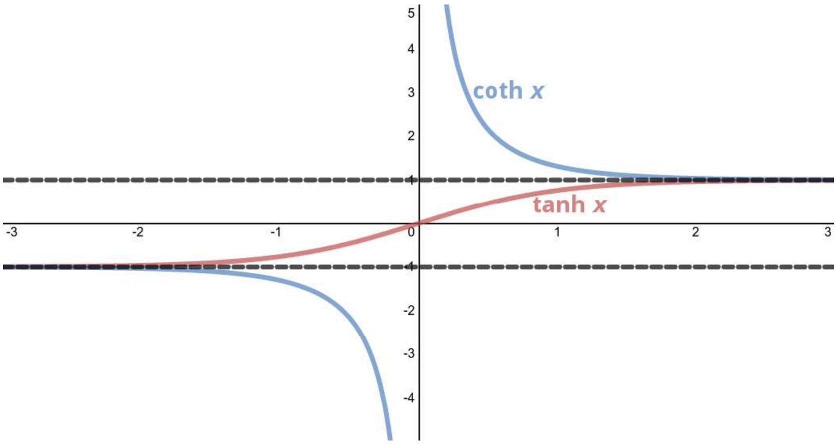

Figure 2.6: Plot of $\tanh x$ and $\coth x$.

Hyperbolic trigonometric identities	
$\cosh^2 x - \sinh^2 x = 1$	$\coth^2 x - 1 = \operatorname{csch}^2 x$
$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$	$\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$	$\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$
$\cosh 2x = \cosh^2 x + \sinh^2 x$	$\cosh x \cosh y = \frac{1}{2}[\cosh(x+y) + \cosh(x-y)]$
$\sinh 2x = 2\sinh x \cosh x$	$\sinh x \sinh y = \frac{1}{2}[\cosh(x+y) - \cosh(x-y)]$
$1 - \tanh^2 x = \operatorname{sech}^2 x$	$\sinh x \cosh y = \frac{1}{2}[\sinh(x+y) + \sinh(x-y)]$

Table 2.2: Some important trigonometric identities.

2. Its image is $\mathbb{R}$ —hence it is surjective.
3. It is monotonic strictly increasing —hence injective.
4. $\log 1 = 0$.
5. $\log(x^a) = a \log x$.
6. $\log(xy) = \log x + \log y$.
7. $\log(x/y) = \log x - \log y$.

2.3 Operations with functions

2.3.1 Algebraic operations

Let $A, B \subset \mathbb{R}$ and consider the two real functions

$$\begin{aligned} f : A &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f(x) \end{aligned} \qquad \begin{aligned} g : B &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = g(x) \end{aligned} \tag{2.10}$$

With these two functions we can perform the following algebraic operations:

(i) **Addition:** If $C = A \cap B$ —where both functions are defined—,

$$\begin{aligned} f + g : C &\longrightarrow \mathbb{R} \\ x &\longrightarrow y = f(x) + g(x) \end{aligned} \tag{2.11}$$