Métodos Matemáticos de Bioingeniería
Grado en Ingeniería Biomédica
Lecture 1

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Introduction and Basic Notions

Outline

1 Introduction and Basic Notions
   - Definition of Vectorial Space
   - Vectors in two and three dimensions: $\mathbb{R}^2$ and $\mathbb{R}^3$
   - Standard basis and parametric equations
   - Examples
Introduction and Basic Notions

Definition of Vectorial Space

Outline

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   - Definition of Vectorial Space
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   - Examples
Definition

A set is a collection of defined objects. In other words a set is given by his elements and has a property that determines whatever is or not in the set.

Examples,

- The set of natural numbers less than 3.
- The set of the prime numbers. \( \mathbb{P} \)

We denote \( x \in S \) when a element \( x \) belongs to the set \( S \). We can describe a set by showing explicitly his elements or we can describe the set using a condition it satisfies:

\[ S = \{ x : x \text{ satisfies condition } P \} \text{. For example, } \]
\[ A = \{ x : x \in \mathbb{N} \text{ and } x < 3 \}. \]
Introduction and Basic Notions

Definition of Vectorial Space

Definition

A correspondence is any rule who associate elements of a set $A$ with elements of a set $B$.

Definition

An application or function is a correspondence where any element of $A$ is associated with an element of $B$ and only one.

We call image or range to the set

$$ f(A) = \{ b \in B : \text{where exists } a, \text{ such as } f(a) = b \}.$$  

An application can be:

- **Injective.** If different elements has different images. So $f(a) = f(a')$ means $a = a'$.

- **Surjective.** If $f(A) = B$. That is that $\forall b \in B$, exists $a \in A$ such as $b = f(a)$.

- **Bijective.** If it is injective and Surjective.
Examples:

- $f(x) = x^2$, $x \in \mathbb{R}$ is just an application. But if we define $x$ only in $\mathbb{N}$ is injective and if we define $x$ in $\mathbb{R}^+$ is bijective.

- $g(x) = e^x$, is injective but not bijective.

- $h(x) = x$, is bijective.

As you see the notion doesn’t depend only on the function $f$ but also of the set where it is defined. If an application is bijective it has inverse $f^{-1} : B \to A$ and is also bijective.

Given two functions $f : A \to B$ and $g : C \to D$, with $f(A) \subseteq C$ his composition, $g \circ f$ is an app from $A \to D$ defined as $g(f(a))$. It is easily verified that in general $g \circ f \neq f \circ g$ but is associative. Try for example: $f(x) = x^2$ and $g(x) = x/\sqrt{x^2 + 1}$.
Definition of Vectorial Space

(a) \( y = e^x \)  

(b) \( y = x \)  

(c) \( y = x^2 \)  

(d) Composition: \( f \circ g \) and \( g \circ f \)

Figura: Examples of functions.
To define a **Vectorial Space** mathematically, we would need to introduce the notion of *group*, *ring*, *field* and more. It is not the goal of this course.

It would look like,

*Let be $\mathbb{K}$ a field and $V$ a non empty set, then $V$ is a vectorial space over $\mathbb{K}$ if...*

It is not very tempting right?

The branches of maths that studies this are called *linear algebra* and *algebraic structures*. As a curios comment, for mathematician a vectorial space is more than our geometrically intuitive space $\mathbb{R}^2$ or $\mathbb{R}^3$. Mathematically, the space of all matrices $\mathbb{M}_{m \times n}$ or the space of the group of all real function are also Vectorial Spaces.
Definition

For us a vectorial space will be $\mathbb{R}^n$ for $n = 1, 2, 3, ...$ associated with two operations defined over the set $\mathbb{R}^n$: the sum and the scalar multiplication.

Figura: $\mathbb{R}^2$ and $\mathbb{R}^3$. 
Introduction and Basic Notions

Vectors in two and three dimensions: $\mathbb{R}^2$ and $\mathbb{R}^3$

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1. Introduction and Basic Notions
   - Definition of Vectorial Space
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Introduction and Basic Notions

Vectors in two and three dimensions: $\mathbb{R}^2$ and $\mathbb{R}^3$

Notation

- We will use **boldface** letters to denote vectors
  
  \[ \mathbf{a} = (a_1, a_2) \in \mathbb{R}^2 \quad \text{or} \quad \mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3 \]
  
- We will refer to single real numbers as **scalars**, $a \in \mathbb{R}$.

Definition

- A **vector** in $\mathbb{R}^2$ is an ordered pair of real numbers
  
  \[ (a_1, a_2) \in \mathbb{R}^2, \quad \text{e.g.,} \quad (\pi, 17) \in \mathbb{R}^2 \]
  
- A **vector** in $\mathbb{R}^3$ is an ordered triple of real numbers
  
  \[ (a_1, a_2, a_3) \in \mathbb{R}^3, \quad \text{e.g.,} \quad (\pi, e, \sqrt{2}) \in \mathbb{R}^3 \]
Introduction and Basic Notions

Vectors in two and three dimensions: $\mathbb{R}^2$ and $\mathbb{R}^3$

Algebraic and Geometric Perspectives

- The Notions of a vector is fundamental for calculus of several variables.
- There are always two points of view: algebraic (above definition) and geometric (visual interpretation).
- Both perspectives are necessary in order to solve problems effectively.
Introduction and Basic Notions

Vectors in two and three dimensions: $\mathbb{R}^2$ and $\mathbb{R}^3$

Definition

- Two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ are **equal** if their corresponding components are equal
  
  \[
  a_1 = b_1 \\
  a_2 = b_2 \\
  a_3 = b_3
  \]

- The same definition holds for vectors in $\mathbb{R}^2$.

Example

- Vectors $\mathbf{a} = (1, 2)$ and $\mathbf{b} = (\frac{3}{3}, \frac{6}{3})$ are equal in $\mathbb{R}^2$.
- Vectors $\mathbf{c} = (1, 2, 3)$ and $\mathbf{d} = (2, 3, 1)$ are not equal in $\mathbb{R}^3$. 
Introduction and Basic Notions

Vectors in two and three dimensions: $\mathbb{R}^2$ and $\mathbb{R}^3$

Definition: Vector Addition

Let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ be two vectors in $\mathbb{R}^3$

The vector sum $a + b$ is the vector in $\mathbb{R}^3$ obtained via componentwise addition:

$$a + b = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

Example

$$(0, 1, 3) + (7, -2, 10) = (7, -1, 13) \text{ in } \mathbb{R}^3$$
$$(1, 1) + (\pi, \sqrt{2}) = (1 + \pi, 1 + \sqrt{2}) \text{ in } \mathbb{R}^2$$
Vectors in two and three dimensions: $\mathbb{R}^2$ and $\mathbb{R}^3$

### Sum properties:

#### Properties of Vector Addition

1. **Commutativity**: $a + b = b + a$ for all $a, b \in \mathbb{R}^3$

2. **Associativity**: $a + (b + c) = (a + b) + c$ for all $a, b, c \in \mathbb{R}^3$

3. **Zero Vector** or neutral element: a special vector $0 = (0, 0, 0)$ with the property that $a + 0 = 0 + a = a$ for all $a \in \mathbb{R}^3$. 
Now we define the scalar multiplication:

**Scalar Multiplication**

- Let \( \mathbf{a} = (a_1, a_2, a_3) \) be a vector in \( \mathbb{R}^3 \).
- Let \( k \in \mathbb{R} \) be a scalar (real number).
- The scalar product \( k \mathbf{a} \) is the vector in \( \mathbb{R}^3 \) given by multiplying each component of \( \mathbf{a} \) by \( k \)

\[
ka = (ka_1, ka_2, ka_3)
\]

**Example**

- If \( \mathbf{a} = (2, 0, \sqrt{2}) \) and \( k = 7 \) then \( k \mathbf{a} = (14, 0, 7\sqrt{2}) \).
Properties of Scalar Multiplication

For all vectors \( \mathbf{a} \) and \( \mathbf{b} \) in \( \mathbb{R}^3 \) and scalars \( k \) and \( l \) in \( \mathbb{R} \), we have

1. \((k + l)\mathbf{a} = k\mathbf{a} + l\mathbf{a}\) (distributivity)

2. \(k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}\) (distributivity)

3. \(k(l\mathbf{a}) = (kl)\mathbf{a} = l(k\mathbf{a})\)
Introduction and Basic Notions

Vectors in two and three dimensions: $\mathbb{R}^2$ and $\mathbb{R}^3$

**First Interpretation: Vectors as points**

- A vector $\mathbf{a}$ in $\mathbb{R}^2$ may be thought of as a **point** in plane $\mathbb{R}^2$ and a vector $\mathbf{a}$ in $\mathbb{R}^3$ may be thought of as a **point** in space $\mathbb{R}^3$:

![Diagram of 2D and 3D Cartesian coordinates]

- This interpretation in terms of points has not meaningful geometric interpretation.
Second Interpretation: Vectors as Positions

- We can visualise a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ as an arrow that begins at the origin and ends at the point. We associate a vector with the point where it ends (bijective application). In this way the elements of $\mathbb{R}^n$ are points but also vectors.

- Such a description is often referred to as the position vector of the point $(a_1, a_2)$ or $(a_1, a_2, a_3)$. 

![Diagram showing vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$]
As we usually have been told, vectors have **magnitude** and **direction**:

### Second Interpretation: Vectors as Positions

- We take **magnitude** to mean *length of the arrow*.
- We take **direction** to be the *orientation or sense of the arrow*.

### Note

- There is an exception to this approach, the **zero vector**.
- It just sits at the origin, like a point.
- It has no magnitude and, therefore, an indeterminate direction.
Introduction and Basic Notions

Vectors in two and three dimensions: $\mathbb{R}^2$ and $\mathbb{R}^3$

Second Interpretation: Vectors as Positions

- In physics, not all vectors are represented by arrows having their tails bound to the origin.
- We need “the freedom” to **parallel translate** vectors throughout $\mathbb{R}^2$ and $\mathbb{R}^3$.
- One may represent the vector $\mathbf{a} = (a_1, a_2, a_3)$ by an arrow with its tail at any point.
For this we need the following definition:

**Definition**

A free vector is a vector with an start not necessarily at the origin. Two of this vectors are the same if we can obtain one from another with a movement of translation.

**Note**

The previous intuitive definition of free vectors, can be mathematically defined as a so-called affine space. Not necessary for this course.
Second Interpretation: Vectors as Positions

- If we wish to represent $\mathbf{a} = (a_1, a_2, a_3)$ by an arrow with its tail at the point $(x_1, x_2, x_3)$

$$\begin{align*}
(x_1 + a_1, x_2 + a_2, x_3 + a_3)
\end{align*}$$

- Then, the head of the arrow would be at the point

$$\begin{align*}
(x_1 + a_1, x_2 + a_2, x_3 + a_3)
\end{align*}$$
Introduction and Basic Notions

Vectors in two and three dimensions: $\mathbb{R}^2$ and $\mathbb{R}^3$

Vector Addition: geometric interpretation

- It is the so-called parallelogram law.
- Assume $\mathbf{a}$ and $\mathbf{b}$ are nonparallel vectors drawn with their tails emanating from the same point.
- Then $\mathbf{a} + \mathbf{b}$ may be represented by the arrow that runs along a diagonal of the parallelogram.
Vector Addition: algebraic and geometric

- We check that geometric constructions agree with algebraic definitions.
- Let \( \mathbf{a} = (a_1, a_2) \) and \( \mathbf{b} = (b_1, b_2) \) be two vectors in \( \mathbb{R}^2 \).
- The arrow obtained from the parallelogram law addition is the one whose tail is at the origin and whose head is at the point: 
  \[
P = (a_1 + b_1, a_2 + b_2)
  \]
Scalar multiplication is easier to visualise.

The vector $ka$ may be represented by an arrow whose:
- length is $|k|$ times the length of $a$.
- direction is the same as that of $a$ when $k > 0$ and the opposite when $k < 0$. 

![Diagram showing scalar multiplication]

- $2a$
- $-\frac{3}{2}a$
Vector subtraction: algebraic Notions and geometric visualization

- The **difference** \( \mathbf{a} - \mathbf{b} \) between two vectors is defined as

\[
\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})
\]

- It may be represented by an arrow pointing from the head of \( \mathbf{b} \) toward the head of \( \mathbf{a} \)

Such an arrow is also a diagonal of the parallelogram determined by \( \mathbf{a} \) and \( \mathbf{b} \).
**Definition 1.5: the displacement vector**

Given two points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ in $\mathbb{R}^3$, the **displacement vector** from $P_1$ to $P_2$ is

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$
Introduction and Basic Notions

Vectors in two and three dimensions: $\mathbb{R}^2$ and $\mathbb{R}^3$

Example: position and velocity of a particle

- Suppose a particle in space is at the point $(a_1, a_2, a_3)$.
- Then, the particle has position vector

$$\mathbf{a} = (a_1, a_2, a_3)$$

- Assume that the particle travels with constant velocity

$$\mathbf{v} = (v_1, v_2, v_3) \text{ for } t \text{ seconds}$$

**Which is the particle’s displacement from its original position?**

**Which is its new coordinate position?**
Example: position and velocity of a particle

- The particle’s displacement from its original position is $tv$
- Its new coordinate position is $\mathbf{a} + tv$

![Diagram showing the displacement and new position of a particle](image-url)
Introduction and Basic Notions

Vectors in two and three dimensions: $\mathbb{R}^2$ and $\mathbb{R}^3$

Example 5

- The S.S. Calculus is cruising due south at a rate of 15 knots (nautical miles per hour) with respect to still water.
- However, there is also a current of $5\sqrt{2}$ knots southeast.

What is the total velocity of the ship?

If the ship is initially at the origin and a lobster pot is at position $(20, -79)$, will the ship collide with the lobster pot?
Example 5

Since velocities are vectors, the total velocity of the ship is

\[ \mathbf{v}_1 + \mathbf{v}_2 \]

- \( \mathbf{v}_1 \) is the velocity of the ship with respect to still water
- \( \mathbf{v}_2 \) is the southeast-pointing velocity of the current
Example 5

- We easily know that $v_1 = (0, -15)$
- Since $v_2$ points southeastward, its direction must be along the line $y = -x$
- Therefore, $v_2$ can be written as $v_2 = (v, -v)$, where $v$ is a positive real number.
- By the Pythagorean theorem, if the length of $v_2$ is $5\sqrt{2}$, then
  
  $$v^2 + (-v)^2 = (5\sqrt{2})^2 \Rightarrow 2v^2 = 50 \Rightarrow v = 5 \Rightarrow v_2 = (5, -5)$$
- Hence, the net velocity is
  
  $$(0, -15) + (5, -5) = (5, -20)$$
Example 5

- After 4 hours, therefore, the ship will be at position

\[(0, 0) + 4(5, -20) = (20, -80)\]

- Thus, it will miss the lobster pot.
Example 6

- The theory behind the art of judo is an excellent example of vector addition.
- If two people, one relatively strong and the other relatively weak, have a shoving match, it is clear who will prevail.
- Someone pushing one way with 200 lb of force will succeed in overpowering another pushing the opposite way with 100 lb.
- Indeed, the net force will be 100 lb in the direction in which the stronger person is pushing.
Example

- The weaker participant applies his or her 100 lb of force in a direction only slightly different from that of the stronger.
- He or she will effect a vector sum of length large enough to surprise the opponent.

![Diagram](image)

- This is the basis for essentially all of the throws of judo.
- This is why judo is described as:

  The art of using a person’s strength against himself or herself
1 Introduction and Basic Notions
   - Definition of Vectorial Space
   - Vectors in two and three dimensions: $\mathbb{R}^2$ and $\mathbb{R}^3$
   - Standard basis and parametric equations
   - Examples
Introduction and Basic Notions

Standard basis and parametric equations

**The Standard Basis Vectors in \( \mathbb{R}^2 \)**

- In \( \mathbb{R}^2 \), a special notational role is played by the vectors

  \[ e_1 = \mathbf{i} = (1, 0) \text{ and } e_2 = \mathbf{j} = (0, 1) \]

- In mathematics is more common to use \( e_1, e_2 \) notation and in engineering the \( i, j \) notation. These vector form what is called the standard or **canonical base**.

- They form a basis because any vector \( \mathbf{a} = (a_1, a_2) \) may be written in terms of them via vector addition and scalar multiplication:

  \[ (a_1, a_2) = (a_1, 0) + (0, a_2) = a_1(1, 0) + a_2(0, 1) = a_1 \mathbf{i} + a_2 \mathbf{j} \]

- They are called canonical because is the natural and most common basis. Nevertheless, any two **linear independent** vectors can be a base.
The Standard Basis Vectors in $\mathbb{R}^2$

- Geometrically, there is a straightforward significance of the **standard basis vectors** $\mathbf{i}$ and $\mathbf{j}$.

- An arbitrary vector $\mathbf{a} \in \mathbb{R}^2$ can be decomposed into appropriate vector components along the $x$- and $y$-axes.
The Standard Basis Vectors in $\mathbb{R}^3$

- Analogously, the standard basis in $\mathbb{R}^3$ is

$$i = (1, 0, 0), \quad j = (0, 1, 0) \quad \text{and} \quad k = (0, 0, 1)$$
The Standard Basis Vectors in $\mathbb{R}^3$

- Any vector $\mathbf{a} = (a_1, a_2, a_3)$ may also be written as

$$a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$$
In $\mathbb{R}^2$, straight lines are described by equations of the form

$$y = mx + b$$

or

$$Ax + By = C$$
Straight lines in $\mathbb{R}^3$

- One might expect the same sort of equation for a line in $\mathbb{R}^3$
  
  However, a single such linear equation describes a plane, not a line

- A pair of simultaneous equations in $x$, $y$ and $z$ is required to define a line.
A curve in the plane may be described analytically by points 
\((x, y)\) where
- \(x\) and \(y\) are given as functions of a third independent variable. 
  \(t\)
- variable \(t\) is known as the parameter.

These functions give rise to parametric equations for the curve.

\[
\begin{align*}
x &= f(t) \\
y &= g(t)
\end{align*}
\]
Example

- Consider the set of equations

\[
\begin{align*}
    x &= 2 \cos t \\
    y &= 2 \sin t
\end{align*}
\]

\[0 \leq t < 2\pi\]

- They describe a circle of radius 2, since we may check that

\[x^2 + y^2 = (2 \cos t)^2 + (2 \sin t)^2 = 4 = 2^2\]
Parametric equations of a curve in $\mathbb{R}^3$

- Parametric equations may be used as readily to describe curves in $\mathbb{R}^3$.

- A curve in $\mathbb{R}^3$ is the set of points $(x, y, z)$ whose coordinates $x, y$ and $z$ are each given by a function of $t$,

\[
\begin{align*}
  x &= f(t) \\
  y &= g(t) \\
  z &= h(t)
\end{align*}
\]
The advantages of using parametric equations are twofold:

- First, they offer a uniform way of describing curves in any number of dimensions.
- Second, they allow you to get a dynamic sense of a curve.

Consider the parameter variable $t$ to represent time and imagine that a particle is travelling along the curve with time.
Parametric equations: geometric visualization

- Geometrically we can assign a **direction** to the curve to signify increasing $t$
- Notice the arrow:

![Graph](image-url)
Parametric equations of lines in $\mathbb{R}^n$

As we know from high-school, a line in $\mathbb{R}^2$ or $\mathbb{R}^3$ is uniquely determined by two pieces of geometric information:

1. A vector whose direction is parallel to that of the line.
2. Any particular point lying on the line.
Parametric equations of lines in $\mathbb{R}^n$

We call the vector

$$\mathbf{r} = \overrightarrow{OP}$$

the position vector of $P(x, y, z)$. 
Standard basis and parametric equations

**Parametric equations of lines in** $\mathbb{R}^n$

- $\mathbf{r} = \overrightarrow{OP}$ can be seen as sum of:
  - The position vector $\mathbf{b}$ of the point $P_0$ (i.e., $\overrightarrow{OP_0}$), and
  - A vector parallel to $\mathbf{a}$. 

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**Introduction and Basic Notions**

- Standard basis and parametric equations
Any vector parallel to \( \mathbf{a} \) must be a scalar multiple of \( \mathbf{a} \).

Letting this scalar be the parameter variable \( t \), we have

\[
\mathbf{r} = \overrightarrow{OP} = \overrightarrow{OP_0} + t \mathbf{a}
\]
Proposition

The vector parametric equation for the line through the point $P_0 = (b_1, b_2, b_3)$, whose position vector is

$$\overrightarrow{OP_0} = \mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}$$

and parallel to

$$\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$$

is:

$$\mathbf{r}(t) = \mathbf{b} + t \mathbf{a}$$
Proposition

- Expanding formula \( \mathbf{r}(t) = \mathbf{b} + t\mathbf{a} \)

\[
\mathbf{r}(t) = \overrightarrow{OP} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k} + t(a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k})
= (a_1 t + b_1)\mathbf{i} + (a_2 t + b_2)\mathbf{j} + (a_3 t + b_3)\mathbf{k}
\]

- Let \( P \) has coordinates \((x, y, z)\)

\[
\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}
\]

- Thus, our parametric equations are

\[
\begin{cases}
x = a_1 t + b_1 \\
y = a_2 t + b_2 \\
z = a_3 t + b_3
\end{cases}
\quad t \in \mathbb{R}
\]
These parametric equations work just as well in $\mathbb{R}^n$

We take $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$

The resulting parametric equations are

$$
\begin{align*}
    x_1 &= a_1 t + b_1 \\
    x_2 &= a_2 t + b_2 \\
    &\vdots \\
    x_n &= a_n t + b_n
\end{align*}
\quad t \in \mathbb{R}
$$
Outline

1. Introduction and Basic Notions
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Example

Find the parametric equations of the line through \((1, -2, 3)\) and parallel to the vector \(\pi \mathbf{i} - 3 \mathbf{j} + \mathbf{k}\)

- \(\mathbf{a} = \pi \mathbf{i} - 3 \mathbf{j} + \mathbf{k}\)
- \(\mathbf{b} = \mathbf{i} - 2 \mathbf{j} + 3 \mathbf{k}\)
- \(\mathbf{r}(t) = \mathbf{i} - 2 \mathbf{j} + 3 \mathbf{k} + t(\pi \mathbf{i} - 3 \mathbf{j} + \mathbf{k}) = (1 + \pi t) \mathbf{i} + (-2 - 3t) \mathbf{j} + (3 + t) \mathbf{k}\)
- The parametric equations may be read as

\[
\begin{align*}
x &= \pi t + 1 \\
y &= -3t - 2 \quad t \in \mathbb{R} \\
z &= t + 3
\end{align*}
\]
From Euclidean geometry, two distinct points determine a unique line in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Find the parametric equations of the line through the points $P_0(1, -2, 3)$ and $P_1(0, 5, -1)$.
Example

- We need to find a vector $\mathbf{a}$ parallel to the desired line.
- The vector with tail at $P_0$ and head at $P_1$ is such a vector:

  $$
  \overrightarrow{P_0P_1} = (0 - 1, 5 - (-2), -1 - 3) = -\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}
  $$

- For $\mathbf{b}$, the position vector of a particular point on the line, we have the choice of taking either:

  $$
  \mathbf{b} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \quad \text{or} \quad \mathbf{b} = 5\mathbf{j} - \mathbf{k}
  $$

- Hence, the parametric equations:

  $$
  \begin{align*}
  x &= 1 - t \\
  y &= -2 + 7t \quad t \in \mathbb{R} \\
  z &= 3 - 4t
  \end{align*}
  \quad \text{or} \quad
  \begin{align*}
  x &= -t \\
  y &= 5 + 7t \quad t \in \mathbb{R} \\
  z &= -1 - 4t
  \end{align*}
  $$
Introduction and Basic Notions

Examples

Parametric Equations of Lines Through Two Distinct Points

- Given two arbitrary points $P_0(a_1, a_2, a_3)$ and $P_1(b_1, b_2, b_3)$
- The line joining them has vector parametric equation

$$
r(t) = \overrightarrow{OP_0} + t\overrightarrow{P_0P_1}
$$

- Which gives parametric equations

$$
\begin{align*}
x &= a_1 + (b_1 - a_1)t \\
y &= a_2 + (b_2 - a_2)t \\
z &= a_3 + (b_3 - a_3)t
\end{align*}
$$

Note

Parametric equations for a line (or, more generally, for any curve) are never unique.
From Parametric Equations To Symmetric Form of a Line

- Assume that each $a_i, \ i = 1, 2, 3$ is nonzero.
- One can eliminate the parameter variable $t$ in each equation

$$\begin{cases}
  x = a_1 t + b_1 \\
  y = a_2 t + b_2 \\
  z = a_3 t + b_3
\end{cases} \quad t \in \mathbb{R} \quad \Rightarrow \quad \begin{cases}
  t = \frac{x - b_1}{a_1} \\
  t = \frac{y - b_2}{a_2} \\
  t = \frac{z - b_3}{a_3}
\end{cases} \quad t \in \mathbb{R}
$$

- Thus, the **symmetric form** is

$$\frac{x - b_1}{a_1} = \frac{y - b_2}{a_2} = \frac{z - b_3}{a_3}$$
Example 4

- The first set of parametric equations give rise to the corresponding symmetric form

\[
\begin{align*}
    x &= 1 - t \\
    y &= -2 + 7t \\
    z &= 3 - 4t
\end{align*}
\]

\[
\begin{align*}
    t &= \frac{x - 1}{-1} \\
    t &= \frac{y + 2}{7} \\
    t &= \frac{z - 3}{-4}
\end{align*}
\]
**Example**

Find where the line with parametric equations

\[
\begin{align*}
x &= t + 5 \\
y &= -2t - 4 \quad t \in \mathbb{R} \\
z &= 3t + 7
\end{align*}
\]

intersects the plane \(3x + 2y - 7z = 2\)

- We must locate the point of intersection.
- One way is to find what value of the parameter \(t\) gives a point on the line that also lies in the plane.
Example

Find where the line with parametric equations

\[
\begin{align*}
    x &= t + 5 \\
    y &= -2t - 4 \quad t \in \mathbb{R} \\
    z &= 3t + 7
\end{align*}
\]

intersects the plane \(3x + 2y - 7z = 2\)

- This is accomplished by substituting the parametric values for \(x, y,\) and \(z\) from the line into the equation for the plane.

\[
3(t + 5) + 2(-2t - 4) - 7(3t + 7) = 2
\]

- Solving the equation for \(t\), we find that \(t = -2\).
Example

Find where the line with parametric equations

\[
\begin{align*}
  x &= t + 5 \\
  y &= -2t - 4 \quad t \in \mathbb{R} \\
  z &= 3t + 7
\end{align*}
\]

intersects the plane \(3x + 2y - 7z = 2\).

- Setting \(t\) equal to \(-2\) in the parametric equations for the line yields the point \((3, 0, 1)\).
- Point \((3, 0, 1)\), indeed, lies in the plane as well.

How should we do this if we start with the symmetric form of the line?
Example 6

Determine whether and where the two lines intersect

\[
\begin{align*}
    x &= t + 1 \\
    y &= 5t + 6 \\
    z &= -2t \\
\end{align*}
\quad t \in \mathbb{R}
\quad \text{and}
\quad \begin{align*}
    x &= 3t - 3 \\
    y &= t \\
    z &= t + 1 \\
\end{align*}
\quad t \in \mathbb{R}
\]

We must be able to find \( t_1 \) and \( t_2 \) so that, by equating the respective parametric expressions for \( x, y \) and \( z \) we have

\[
\begin{align*}
    t_1 + 1 &= 3t_2 - 3 \\
    5t_1 + 6 &= t_2 \\
    -2t_1 &= t_2 + 1 \\
\end{align*}
\quad t \in \mathbb{R}
\]
Example 6

Determine whether and where the two lines intersect

\[
\begin{align*}
x &= t + 1 \\
y &= 5t + 6 \quad t \in \mathbb{R} \quad \text{and} \\
z &= -2t
\end{align*}
\]

\[
\begin{align*}
x &= 3t - 3 \\
y &= t \\
z &= t + 1
\end{align*}
\]

Using the last two equations

\[
t_2 = 5t_1 + 6 = -2t_1 - 1 \implies t_1 = -1
\]
Example 6

Determine whether and where the two lines intersect

\[
\begin{align*}
x &= t + 1 \\
y &= 5t + 6 & \quad t \in \mathbb{R} \quad \text{and} \quad x &= 3t - 3 \\
z &= -2t \\
y &= t & \quad t \in \mathbb{R} \\
z &= t + 1
\end{align*}
\]

Using \( t_1 = -1 \) in the second equation, we find that \( t_2 = 1 \).

Note that the values \( t_1 = -1 \) and \( t_2 = 1 \) also satisfy the first equation.
Example 6

Determine whether and where the two lines

\[
\begin{align*}
  x &= t + 1 \\
  y &= 5t + 6 & t \in \mathbb{R} \\
  z &= -2t
\end{align*}
\]

and

\[
\begin{align*}
  x &= 3t - 3 \\
  y &= t \\
  z &= t + 1
\end{align*}
\]

intersect

Setting \( t = 1 \) in the set of parametric equations for the first line gives the desired intersection point, namely, \((0, 1, 2)\).
Example 7

- Assume a wheel rolls along a flat surface without slipping.
- A point on the rim of the wheel traces a curve called a cycloid.

Vector geometry makes it relatively easy to find parametric equations.
Example 7

Suppose that the wheel has radius $a$

Suppose that coordinates in $\mathbb{R}^2$ are chosen so that the point of interest on the wheel is initially at the origin
After the wheel has rolled through a central angle of $t$ radians, the situation is as shown in figure.
The parametric equations are

\[
\begin{align*}
  x &= a(t - \sin t) \\
  y &= a(1 - \cos t)
\end{align*}
\quad t \in \mathbb{R}
\]