Métodos Matemáticos de Bioingeniería

Grado en Ingeniería Biomédica

Lecture 6

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Outline

Outline

1. The Derivative - Section 2.3
   - Partial derivatives
   - Differenciability
   - Matrix notation and differentiability in $\mathbb{R}^n$

Derivative of a scalar-valued function of one variable

- Let $F : X \subseteq \mathbb{R} \to \mathbb{R}$ be a scalar-valued function of one variable.
- The derivative of F at a number $a \in X$ is

$$F'(a) = \lim_{h \to 0} \frac{F(a+h) - F(a)}{h}$$

F is said to be **differentiable** at a
when the limit in this equation exists

Definition 3.1: Partial Function

- Suppose $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is a scalar-valued function of n variables.
- Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ denote a point of $\mathbb{R}^n$.
- A partial function F with respect to the variable x_i is a one-variable function obtained from f by holding all variables constant except x_i.
- That is, we set x_j equal to a constant a_j for $j \neq i$.
- Then the partial function in x_i is defined by

$$F(x_i) = f(a_1, a_2, \ldots, x_i, \ldots, a_n)$$

Remark

- We usually do not replace the x_j's ($j \neq i$) by constants.
- Instead, we make a **mental note**.

Example 1

- Consider the function

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

- Then the partial functions with respect to x are given by

$$F(x) = f(x, a_2) = \frac{x^2 - a_2^2}{x^2 + a_2^2}$$

  where a_2 may be any constant.

- If, for example, $a_2 = 0$, then the partial function is

$$F(x) = f(x, 0) = \frac{x^2}{x^2} \equiv 1$$

Example 1

- Consider the function

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

- Then the partial functions with respect to x are given by

$$F(x) = f(x,0) = \frac{x^2}{x^2} \equiv 1$$

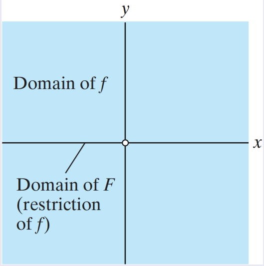
- Geometrically, this partial function is the restriction of f to the horizontal line $y = 0$.
- Since the origin is not in the domain of f, value 0 should not be taken to be in the domain of F.

Example 1

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

$$F(x) = f(x, a_2) = \frac{x^2 - a_2^2}{x^2 + a_2^2}$$

- The function f is defined on $\mathbb{R}^2 - \{(0,0)\}$
- Its partial function F is defined on the x-axis minus the origin.



Definition 2.2: Partial Derivative

- The partial derivative of f with respect to x_i is the (ordinary) derivative of the partial function with respect to x_i.
- In the notation of Definition 3.1, partial derivative with respect to x_i is

$$F'(x_i)$$

- **Standard notations** for the partial derivative of f with respect to x_i are:

$$\frac{\partial f(x_1, \ldots, x_n)}{\partial x_i}, \quad D_{x_i} f(x_1, \ldots, x_n) \quad \text{and} \quad f_{x_i}(x_1, \ldots, x_n)$$

or

$$\frac{\partial f}{\partial x_i}, \quad D_{x_i} f \quad \text{and} \quad f_{x_i}$$

Definition 2.2: Analytic and Geometric interpretation

- Symbolically, we have

$$\frac{\partial f}{\partial x_i} = \lim_{h\to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$

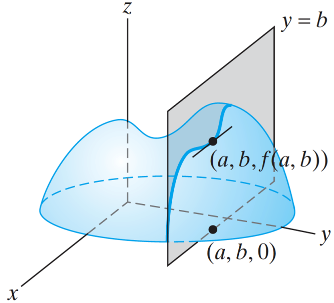
- The partial derivative is the (instantaneous) rate of change of f when all variables, except the specified one, are held fixed.
- In the case where f is a (scalar-valued) function of two variables we can consider,

$$\frac{\partial f}{\partial x}(a, b) \quad \text{and} \quad \frac{\partial f}{\partial y}(a, b)$$

Geometric Interpretation of Partial Derivatives in $\mathbb{R}^2$

$$\frac{\partial f}{\partial x}(a, b)$$

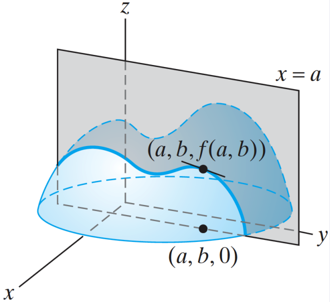
- Geometrically it is the slope at the point $(a, b, f(a, b))$ of the curve obtained by intersecting
  - The surface $z = f(x, y)$ with
  - The plane $y = b$



Geometric Interpretation of Partial Derivatives in $\mathbb{R}^2$

$$\frac{\partial f}{\partial y}(a, b)$$

- Geometrically it is the slope at the point $(a, b, f(a, b))$ of the curve obtained by intersecting
  - The surface $z = f(x, y)$ with
  - The plane $x = a$



Example 2a

- Let

$$f(x, y) = x^2 y + \cos(x + y)$$

- Then, if we imagine y to be a constant throughout the differentiation process, we have,

$$\frac{\partial f}{\partial x} = 2xy - \sin(x + y)$$

- If we imagine x to be a constant,

$$\frac{\partial f}{\partial y} = x^2 - \sin(x + y)$$

Example 2b

- Let

$$g(x,y) = \frac{xy}{(x^2+y^2)}$$

- Then

$$\begin{aligned} g_x(x,y) &= \frac{(x^2+y^2)y - xy(2x)}{(x^2+y^2)^2} = \frac{y(y^2-x^2)}{(x^2+y^2)^2} \\ g_y(x,y) &= \frac{(x^2+y^2)x - xy(2y)}{(x^2+y^2)^2} = \frac{x(x^2-y^2)}{(x^2+y^2)^2} \end{aligned}$$

- Note that neither g nor its partial derivatives are defined at point $(0,0)$.

Example 3

- Occasionally, it is necessary to appeal explicitly to limits to evaluate partial derivatives.
- Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} \frac{3x^2y - y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

1. For $(\mathbf{x}, \mathbf{y}) \neq (\mathbf{0}, \mathbf{0})$, we have

$$\frac{\partial f}{\partial x} = \frac{8xy^3}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{3x^4 - 6x^2y^2 - y^4}{(x^2+y^2)^2}$$

Example 3

- Occasionally, it is necessary to appeal explicitly to limits to evaluate partial derivatives:
- Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} \frac{3x^2y - y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

2. For $(x,y) = (0,0)$, we return to Definition 3.2

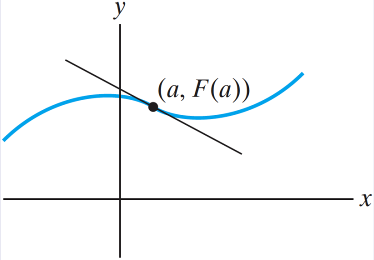
$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{-h-0}{h} = \lim_{h\to 0} -1 = -1$$

Outline

Tangency for scalar-valued functions of one variable

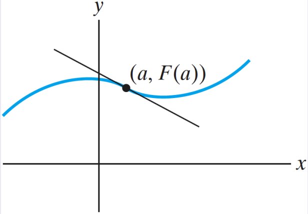
- Let $F : X \subseteq \mathbb{R} \to \mathbb{R}$ be a scalar-valued function of one variable.
- F is differentiable at a number $a \in X$ if the graph of the curve $y = F(x)$ has a tangent line at the point $(a, F(a))$.



- This tangent line is given by the equation:

$$y = F(a) + F'(a)(x - a)$$

Tangency for scalar-valued functions of one variable



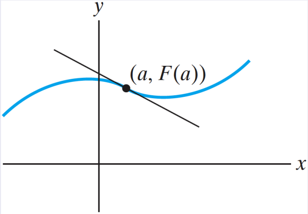
$$y = F(a) + F'(a)(x - a)$$

- If we define the function $H(x)$ to be

$$H(x) = F(a) + F'(a)(x - a)$$

- Then H has two properties:
  1. $H(a) = F(a)$ The line defined by $y = H(x)$ passes through the point $(a, F(a))$

Tangency for scalar-valued functions of one variable



$$y = F(a) + F'(a)(x - a)$$

- If we define the function $H(x)$ to be

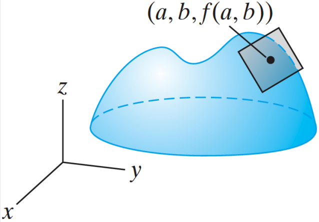
$$H(x) = F(a) + F'(a)(x - a)$$

- Then H has two properties:
  2. $H'(a) = F'(a)$

     the line defined by $y = H(x)$ has the same slope at $(a, F(a))$ as the curve defined by $y = F(x)$

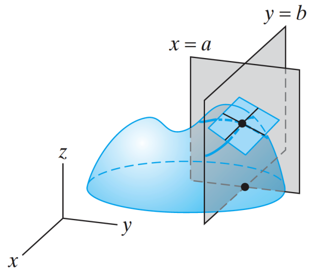
Tangency for scalar-valued functions of two variables

- Suppose $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a scalar-valued function of two variables.
- Suppose X is open in $\mathbb{R}^2$ so the graph of f is a surface.
- What should the tangent plane to the graph of $z = f(x, y)$ at the point $(a, b, f(a, b))$ be ?



Tangency for scalar-valued functions of two variables

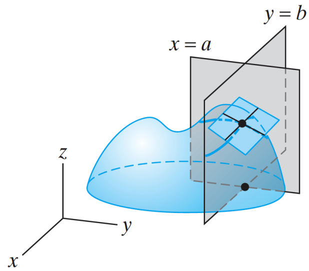
1. The partial derivative $f_x(a, b)$ is the slope of the line tangent at the point $(a, b, f(a, b))$ to the curve obtained by intersecting the surface $z = f(x, y)$ with the plane $y = b$



- If we travel along this tangent line, then for every unit change in the positive x-direction, there is a change of $f_x(a, b)$ units in the z-direction.

Tangency for scalar-valued functions of two variables

1. The partial derivative $f_x(a, b)$ is the slope of the line tangent at the point $(a, b, f(a, b))$ to the curve obtained by intersecting the surface $z = f(x, y)$ with the plane $y = b$

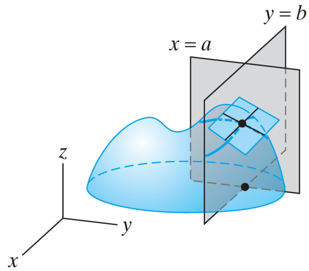


- The tangent line is given in vector parametric form as:

$$\mathbf{l}_1(t) = (a, b, f(a, b)) + t(1, 0, f_x(a, b))$$

Tangency for scalar-valued functions of two variables

1. The partial derivative $f_x(a,b)$ is the slope of the line tangent at the point $(a,b,f(a,b))$ to the curve obtained by intersecting the surface $z=f(x,y)$ with the plane $y=b$

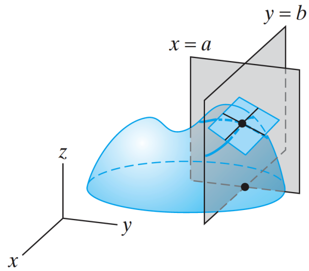


- Thus, a vector parallel to this tangent line is

$$\mathbf{u} = \mathbf{i} + f_x(a,b)\mathbf{k}$$

Tangency for scalar-valued functions of two variables

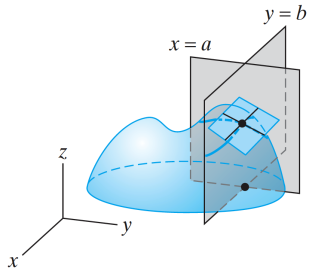
2. Analogously, the partial derivative $f_y(a,b)$ is the slope of the line tangent at the point $(a,b,f(a,b))$ to the curve obtained by intersecting the surface $z=f(x,y)$ with the plane $x=a$.



- If we travel along this tangent line, then for every unit change in the positive y-direction, there is a change of $f_y(a,b)$ units in the z-direction.

Tangency for scalar-valued functions of two variables

2. The partial derivative $f_y(a,b)$ is the slope of the line tangent at the point $(a, b, f(a,b))$ to the curve obtained by intersecting the surface $z = f(x,y)$ with the plane $x = a$.

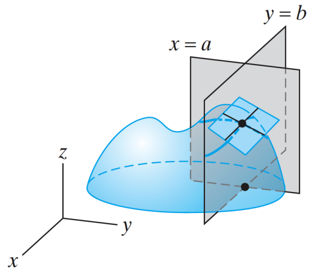


- The tangent line is given in vector parametric form as:

$$\mathbf{l}_2(t) = (a, b, f(a,b)) + t(0, 1, f_y(a,b))$$

Tangency for scalar-valued functions of two variables

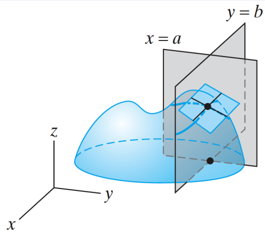
2. The partial derivative $f_y(a, b)$ is the slope of the line tangent at the point $(a, b, f(a, b))$ to the curve obtained by intersecting the surface $z = f(x, y)$ with the plane $x = a$



- Thus, a vector parallel to this tangent line is:

$$\mathbf{v} = \mathbf{j} + f_y(a, b)\mathbf{k}$$

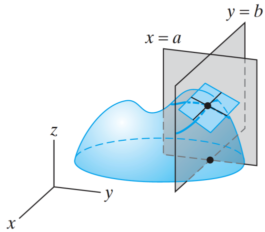
Tangency for scalar-valued functions of two variables



$$\mathbf{l}_1(t) = (a, b, f(a, b)) + t(1, 0, f_x(a, b)), \quad \mathbf{u} = \mathbf{i} + f_x(a, b)\mathbf{k}$$
$$\mathbf{l}_2(t) = (a, b, f(a, b)) + t(0, 1, f_y(a, b)), \quad \mathbf{v} = \mathbf{j} + f_y(a, b)\mathbf{k}$$

- Both of the tangent lines must be contained in the plane tangent to $z = f(x, y)$ at $(a, b, f(a, b))$, if one exists.
- A vector **n** normal to the tangent plane must be perpendicular to both **u** and **v**.

Tangency for scalar-valued functions of two variables



$$\mathbf{l}_1(t) = (a, b, f(a,b)) + t(1, 0, f_x(a,b)), \quad \mathbf{u} = \mathbf{i} + f_x(a,b)\mathbf{k}$$
$$\mathbf{l}_2(t) = (a, b, f(a,b)) + t(0, 1, f_y(a,b)), \quad \mathbf{v} = \mathbf{j} + f_y(a,b)\mathbf{k}$$

- A vector n normal to the tangent plane must be perpendicular to both **u** and **v**
- Therefore, we may take **n** to be:

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = -f_x(a,b)\mathbf{i} - f_y(a,b)\mathbf{j} + \mathbf{k}$$

Tangency for scalar-valued functions of two variables

We have the normal vector and a point of the tangent plane.

$$\mathbf{n} \quad = \quad \mathbf{u} \times \mathbf{v} = -f_x(a,b)\mathbf{i} - f_y(a,b)\mathbf{j} + \mathbf{k}; \quad P = (a, b, f(a,b))$$

- So, the **equation for the tangent plane** through $(a, b, f(a,b))$ with normal $\mathbf{n}$ is

$$(-f_x(a,b), -f_y(a,b), 1) \cdot (x-a, y-b, z-f(a,b)) = 0$$

or

$$-f_x(a,b)(x-a) - f_y(a,b)(y-b) + z - f(a,b) = 0$$

Theorem 3.3

- If the graph of $z = f(x, y)$ has a tangent plane at $(a, b, f(a, b))$ then that tangent plane has equation

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- Let define the function $h(x, y)$ to be

$$h(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- Then h has the following properties
  1. $h(a, b) = f(a, b)$

     The values of h and f are the same at (a, b)
  2. $\frac{\partial h}{\partial x}(a, b) = \frac{\partial f}{\partial x}(a, b)$ and $\frac{\partial h}{\partial y}(a, b) = \frac{\partial f}{\partial y}(a, b)$

     Partial derivatives of h and f are the same at (a, b)

Definition 3.4: Differentiability

- Let X be open in $\mathbb{R}^2$.
- Let $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a scalar-valued function of two variables.
- We say that f is differentiable at $(a, b) \in X$ if
  - The partial derivatives $f_x(a, b)$ and $f_y(a, b)$ exist , and
  - The function

$$h(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is a good linear approximation to f near (a, b):

$$\text{lím}_{(x,y)\to(a,b)} \frac{f(x, y) - h(x, y)}{\|(x, y) - (a, b)\|} = 0$$

Definition 3.4: Differentiability

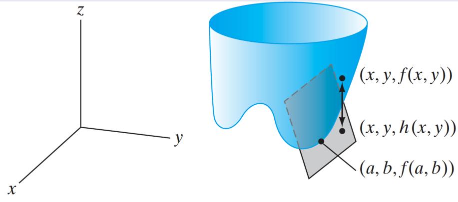
- Mathematically is not necessary to suppose the partial derivatives exists. It is enough to say that f is differentiable if exists a **linear function** h that:

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)-h(x,y)}{\|(x,y)-(a,b)\|}=0$$

- If f is differentiable at (a,b), then the equation $z=h(x,y)$ defines the tangent plane to the graph of f at the point $(a,b,f(a,b))$.

If f is differentiable at all points of its domain, then we say that f is **differentiable**

Definition 3.4: Differentiability

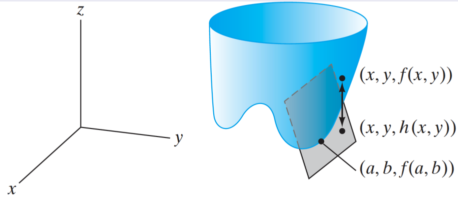


$$h(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$\lim_{(x,y)\to(a,b)} \frac{f(x, y) - h(x, y)}{\|(x, y) - (a, b)\|} = 0$$

- To say that $z = f(x, y)$ has a tangent plane at $(a, b, f(a, b))$ is to say that f is differentiable at (a, b).

Definition 3.4: Differentiability



$$h(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$\lim_{(x,y)\to(a,b)} \frac{f(x, y) - h(x, y)}{\|(x, y) - (a, b)\|} = 0$$

- The **vertical distance** between the graph of f and the tangent plane $z = h(x, y)$ **must approach zero faster** than the point (x, y) approaches (a, b).

The limit condition can be difficult to apply in practice. Hence, this theorem could be useful.

Theorem 3.5

- Suppose X is open in $\mathbb{R}^2$.
- If $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives in a neighbourhood of (a, b) in X, then f is differentiable at (a, b).

Example 6

- Let $f(x, y) = x^2 + 2y^2$
- Then

$$\frac{\partial f}{\partial x} = 2x \quad \text{and} \quad \frac{\partial f}{\partial y} = 4y$$

Example 6

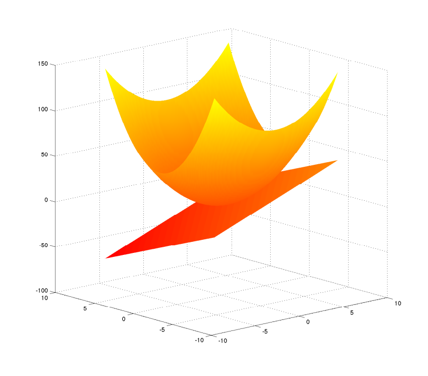
- Let $f(x,y) = x^2 + 2y^2$
- Thus, Theorem 3.5 implies that f is differentiable everywhere.
- The surface $z = x^2 + 2y^2$ must have a tangent plane at every point ,
- At the point $(2,-1)$, for example, this tangent plane is given by the equation:

$$z = 6 + 4(x-2) - 4(y+1) \quad \text{or} \quad 4x - 4y - z = 6$$

Example 6

- Let $f(x,y) = x^2 + 2y^2$
- At the point $(2,-1)$, for example, this tangent plane is

$$z = 6 + 4(x-2) - 4(y+1) \quad \text{or} \quad 4x - 4y - z = 6$$



Theorem 3.6

- If $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a, b), then it is continuous at (a, b).

Example 7

- Let the function $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^4+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

- The function f is not continuous at the origin, since

$$\lim_{(x,y)\to(0,0)} f(x,y) \quad \text{does not exist}$$

- However, f is continuous everywhere else in $\mathbb{R}^2$.
- By Theorem 3.6, f cannot be differentiable at the origin.

Theorem 3.6

- If $f : X \subseteq \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a, b), then it is continuous at (a, b)

Example 7

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^4+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

- Nonetheless, the partial derivatives of f do exist at the origin since the partial functions are constant

$$\begin{aligned} f(x,0) &= \frac{0}{x^4+0} \equiv 0 \Rightarrow \frac{\partial f}{\partial x}(0,0) = 0 \\ f(0,y) &= \frac{0}{0+y^4} \equiv 0 \Rightarrow \frac{\partial f}{\partial y}(0,0) = 0 \end{aligned}$$

The existence of partial derivatives alone is not enough

Differentiation Remarks

We have the following hierarchy:

Continuous partials $\Rightarrow$ *Differentiable* $\Rightarrow$ *Continuous function and Partials exist (but not necessary continuous)*

The inverse implications doesn't follow. To prove it you can take this example for the first:

$$f(x) = x^2 \sin(1/x), \; f(0) = 0.$$

And for the second:

$$f(x, y) = xy/\sqrt{x^2 + y^2}, \; f(0, 0) = 0$$

(continuous function).

Outline

Generalisation of scalar-valued function in $\mathbb{R}^n$

- Let X be open in $\mathbb{R}^n$.
- Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function.
- Let $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in X$.
- We say that f is differentiable at $\mathbf{a}$ if
  - All the partial derivatives $f_{x_i}(a), i = 1, \ldots, n$, exist, and
  - The function $h : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\begin{aligned} h(\mathbf{x}) &= f(\mathbf{a}) + f_{x_1}(\mathbf{a})(x_1 - a_1) + f_{x_2}(\mathbf{a})(x_2 - a_2) \\ &\quad + \cdots + f_{x_n}(\mathbf{a})(x_n - a_n) \end{aligned}$$

  is a good linear approximation to f near $\mathbf{a}$

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{f(\mathbf{x}) - h(\mathbf{x})}{\|\mathbf{x} - \mathbf{a}\|} = 0$$

Matrix Notation and Gradient

- Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function.
- We define the gradient of f to be the vector ,

$$\nabla f(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

- Consequently,

$$\nabla f(\mathbf{a}) = (f_{x_1}(\mathbf{a}), f_{x_2}(\mathbf{a}), \ldots, f_{x_n}(\mathbf{a}))$$

- Alternatively, we can use matrix notation and define the derivative of f at $\mathbf{a}$.

The **derivative** of f at $\mathbf{a}$, $Df(\mathbf{a})$, is the row matrix whose entries are the components of $\nabla f(\mathbf{a})$

Matrix Notation and Gradient

- Hence, vector notation allows us to rewrite equation

$$\begin{aligned} h(\mathbf{x}) &= f(\mathbf{a}) + f_{x_1}(\mathbf{a})(x_1 - a_1) + f_{x_2}(\mathbf{a})(x_2 - a_2) \\ &\quad + \cdots + f_{x_n}(\mathbf{a})(x_n - a_n) \end{aligned}$$

- Compactly

$$h(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$$

- Thus, to say that h is a good linear approximation to f near $\mathbf{a}$ means that

$$\text{lím}_{\mathbf{x}\to\mathbf{a}} \frac{f(\mathbf{x}) - [f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})]}{\|(\mathbf{x} - \mathbf{a})\|} = 0$$

Matrix of partial derivatives for vector-valued functions

- Let X be open in $\mathbb{R}^n$.
- Let $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function of n variables.
- We define the matrix of partial derivatives of $\mathbf{f}$, denoted $D\mathbf{f}$ and called the **Jacobian Matrix**. This is the $m \times n$ matrix whose ijth entry is:

$$\frac{\partial f_i}{\partial x_j}$$

where $f_i : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is the ith component function of $\mathbf{f}$.

$$D\mathbf{f}(x_1, x_2, \ldots, x_n) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Matrix of partial derivatives for vector-valued functions

$$D\mathbf{f}(x_1, x_2, \ldots, x_n) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

- The ith row of $D\mathbf{f}$ is nothing more than Df_i.
- The entries of Df_i are precisely the components of the gradient vector ∇f_i.
- In the case where $m = 1$, $\nabla\mathbf{f}$ and $\mathbf{Df}$ mean exactly the **same thing**.

Example 9

- Suppose $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$ is given by

$$\mathbf{f}(x, y, z) = (x \cos y + z, xy)$$

- Then we have,

$$\mathbf{f}(x, y, z) = (f_1(x, y, z), f_2(x, y, z))$$

where,

$$\begin{aligned} f_1(x, y, z) &= x \cos y + z \\ f_2(x, y, z) &= xy \end{aligned}$$

- Thus,

$$D\mathbf{f}(x, y, z) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos y & -x \sin y & 1 \\ y & x & 0 \end{bmatrix}$$

Definition 3.8: Grand Definition of Differentiability

- Let X be open in $\mathbb{R}^n$ and let $\mathbf{a} \in X$
- Let $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$
- We say that $\mathbf{f}$ is differentiable at $\mathbf{a}$ if
  - $D\mathbf{f}(\mathbf{a})$ exists, and
  - The function $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$\mathbf{h}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + D\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

is a good linear approximation to $\mathbf{f}$ near $\mathbf{a}$

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{\|\mathbf{f}(\mathbf{x}) - \mathbf{h}(\mathbf{x})\|}{\|\mathbf{x} - \mathbf{a}\|} = \lim_{\mathbf{x} \to \mathbf{a}} \frac{\|\mathbf{f}(\mathbf{x}) - [\mathbf{f}(\mathbf{a}) + D\mathbf{f}(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})]\,\|}{\|(\mathbf{x} - \mathbf{a})\|} = 0$$

- The term $D\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a})$ should be interpreted as the product of the $m \times n$ matrix $D\mathbf{f}(\mathbf{a})$ and the $n \times 1$ column matrix

$$\begin{bmatrix} x_1 - a_1 & x_2 - a_2 & \cdots & x_n - a_n \end{bmatrix}^T$$

Theorem 3.9

If $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{a}$, then it is continuous at $\mathbf{a}$.

Theorem 3.10

- Suppose $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ such that, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, all

$$\frac{\partial f_i}{\partial x_j}$$

  - Exist, and
  - Are continuous in a neighbourhood of $\mathbf{a}$ in X
- Then, $\mathbf{f}$ is differentiable at $\mathbf{a}$.

Theorem 3.11

A function $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{a} \in X$

(in the sense of Definition 3.8)

if and only if

Each of its component functions $f_i : X \subseteq \mathbb{R}^n \to \mathbb{R}, i = 1, ..., m$, is differentiable at $\mathbf{a}$

(in the sense of Definition 3.7)

Example 10

- Suppose $\mathbf{g} : \mathbb{R}^3 - \{(0,0,0)\} \to \mathbb{R}^3$ is given by

$$\mathbf{g}(x,y,z) = \left(\frac{3}{x^2+y^2+z^2}, xy, xz \right)$$

- Then we have

$$\mathbf{g}(x,y,z) = (g_1(x,y,z), g_2(x,y,z), g_3(x,y,z))$$

where

$$\begin{aligned} g_1(x,y,z) &= \frac{3}{x^2+y^2+z^2} \\ g_2(x,y,z) &= xy \\ g_3(x,y,z) &= xz \end{aligned}$$

Example 10

$$\begin{aligned}
\mathbf{g}(x,y,z) &= \left(\frac{3}{x^2+y^2+z^2}, xy, xz\right)\\
g_1(x,y,z) &= \frac{3}{x^2+y^2+z^2}\\
g_2(x,y,z) &= xy\\
g_3(x,y,z) &= xz
\end{aligned}$$

- Thus

$$D\mathbf{g}(x,y,z)=\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z}\\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix}=\begin{bmatrix} \frac{-6x}{(x^2+y^2+z^2)^2} & \frac{-6y}{(x^2+y^2+z^2)^2} & \frac{-6z}{(x^2+y^2+z^2)^2}\\ y & x & 0\\ z & 0 & x \end{bmatrix}$$

Example 10

$$D\mathbf{g}(x,y,z) = \begin{bmatrix} \frac{-6x}{(x^2+y^2+z^2)^2} & \frac{-6y}{(x^2+y^2+z^2)^2} & \frac{-6z}{(x^2+y^2+z^2)^2} \\ y & x & 0 \\ z & 0 & x \end{bmatrix}$$

- Each of the entries of this matrix is continuous over $\mathbb{R}^3 - \{(0,0,0)\}$
- Hence, by Theorem 3.10, $\mathbf{g}$ is differentiable over its entire domain.